Classical/Quantum Dynamics in a Uniform Gravitational Field:

A. Unobstructed Free Fall

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Introduction. Every student of mechanics encounters discussion of the idealized one-dimensional dynamical systems

$$F(x) = m\ddot{x} \quad \text{with} \quad F(x) = \begin{cases} 0 & : \quad \text{FREE PARTICLE} \\ -mg & : \quad \text{PARTICLE IN FREE FALL} \\ -kx & : \quad \text{HARMONIC OSCILLATOR} \end{cases}$$

at a very early point in his/her education, and with the first & last of those systems we are never done: they are—for reasons having little to do with their physical importance—workhorses of theoretical mechanics, traditionally employed to illustrated formal developments as they emerge, one after another. But—unaccountably—the motion of particles in uniform gravitational fields[1] is seldom treated by methods more advanced than those accessible to beginning students.

It is true that terrestrial gravitational forces are so relatively weak that they can—or could until recently—usually be dismissed as irrelevant to the phenomenology studied by physicists in their laboratories.[2] But the free fall

[1] It is to avoid that cumbersome phrase that I adopt the "free fall" locution, though technically *free fall presumes neither uniformity nor constancy of the ambient gravitational field*. If understood in the latter sense, "free fall" lies near the motivating heart of general relativity, which is certainly *not* a neglected subject.

[2] There are exceptions: I am thinking of experiments done several decades ago where neutron diffraction in crystals was found to be sensitive to the value of g. And of experiments designed to determine whether antiparticles "fall up" (they don't).

problem—especially if construed quantum mechanically, with an eye to the *relation between* the quantum physics and classical physics—is itself a kind of laboratory, one in which striking clarity can be brought to a remarkable variety of formal issues. It is those points of general principle that interest me, but some of the technical details (especially those that hinge on properties of the Airy function $\text{Ai}(z)$) are so lovely as to comprise their own reward.

I approach the classical theory of free fall with an eye to the formal needs of the quantum theory, and consider the entire discussion to be merely preparatory to the discussions of the classical/quantum dynamics of a bouncing ball and of some related instances of *obstructed* free fall that will be presented in a series of companion essays.

CLASSICAL DYNAMICS OF FREE FALL

1. What Galileo & Newton have to say about the problem. We will work in one dimension and (because I want to hold both y and z in reserve) consider the Cartesian x-axis to run "up." The free fall problem arises from

$$F(x) = m\ddot{x}$$

when $F(x)$ is in fact an x-independent *constant*, the "weight" of the lofted particle . . . and with that we confront already a conceptual issue, wrapped in an anachronism:

To describe "weight" we might, in general, write

$$\boldsymbol{W} = m_g\boldsymbol{g}$$

where m_g refers to the *gravitational mass* of the particle, and $\boldsymbol{g}$ to the strength of the ambient gravitational field. It becomes therefore possible for Newton (but would have been anachronistic for Galileo) to contemplate equations of motion of the form

$$m_g\boldsymbol{g} = m\ddot{\boldsymbol{x}}$$

But Galileo's Leaning Tower Experiment supplies $m_g = m$ (universally). It is on this basis that we drop the $_g$: $m_g\boldsymbol{g} = m\ddot{\boldsymbol{x}}$ becomes $m\boldsymbol{g} = m\ddot{\boldsymbol{x}}$ becomes $\boldsymbol{g} = \ddot{\boldsymbol{x}}$, which in the 1-dimensional case, if $\boldsymbol{g}$ is a down-directed vector of magnitude g, becomes

$$\ddot{x} + g = 0 \tag{1}$$

We will have quantum mechanical occasion to revisit this topic.

2. Free fall trajectories in spacetime. The general solution of (1) can be described

$$x(t) = a + bt - \tfrac{1}{2}gt^2 \tag{2}$$

where a and b are arbitrary constants (constants of integration). Standardly we associate a and b with *initial data*

$$a \equiv x(0) \equiv x_0 \quad : \quad b \equiv \dot{x}(0) \equiv v_0$$

but other interpretations are sometimes more useful: from stipulated endpoint conditions

$$x_0 = a + bt_0 - \tfrac{1}{2}gt_0^2$$
$$x_1 = a + bt_1 - \tfrac{1}{2}gt_1^2$$

we compute (it's easy by hand if we compute b first, but ask *Mathematica*)

$$\left.\begin{aligned} a &= \frac{x_0t_1 - x_1t_0}{t_1 - t_0} - \tfrac{1}{2}gt_0t_1 \\ b &= \frac{x_1 - x_0}{t_1 - t_0} + \tfrac{1}{2}g(t_0 + t_1) \end{aligned}\right\} \tag{3}$$

giving

$$\begin{aligned} x(t;x_1,t_1;x_0,t_0) = &\left\{\frac{x_0t_1 - x_1t_0}{t_1 - t_0} - \tfrac{1}{2}gt_0t_1\right\} \\ &+ \left\{\frac{x_1 - x_0}{t_1 - t_0} + \tfrac{1}{2}g(t_0 + t_1)\right\}t - \tfrac{1}{2}gt^2 \end{aligned} \tag{4}$$

which checks out: $x(t_0;x_1,t_1;x_0,t_0) = x_0$, $x(t_1;x_1,t_1;x_0,t_0) = x_1$.

3. Translational equivalence in spacetime. Spacetime translation of what we will agree to call the "primitive solution"

$$x = -\tfrac{1}{2}gt^2 \tag{5.1}$$

gives

$$(x - x_0) = -\tfrac{1}{2}g(t - t_0)^2 \tag{5.2}$$

or

$$x = (x_0 - \tfrac{1}{2}gt_0^2) + (gt_0)t - \tfrac{1}{2}gt^2$$

To say the same thing another way: if into (2) we introduce the notation

$$\left.\begin{aligned} b &= gt_0 \\ a &= x_0 - \tfrac{1}{2g}b^2 = x_0 - \tfrac{1}{2}gt_0^2 \end{aligned}\right\} \tag{6}$$

then we achieve (5.2). In short: *every solution is translationally equivalent to the primitive solution*. In this regard the free fall problem is distinguished from both the FREE PARTICLE PROBLEM

> General solution of $\ddot{x} = 0$ reads $x(t) = a + bt$. Take $x(t) = Bt$ to be the primitive solution (B a prescribed constant "velocity"). To bring $a + bt$ to the form Bt one must
> - translate in space and
> - <u>rescale</u> the time coordinate

and the HARMONIC OSCILLATOR

> General solution of $\ddot{x} = -\omega^2 x$ reads $x(t) = a\cos\omega t + (b/\omega)\sin\omega t$. Take $x(t) = (B/\omega)\sin\omega t$ to be the primitive solution (here B is again a prescribed constant "velocity": we have arranged to recover our free particle conventions in the limit $\omega \downarrow 0$). To bring the general solution to primitive form one must
> - translate in time and
> - rescale the space coordinate

REMARK: Some figures here would make everything clear. Figure 1 captures the situation as it relates specifically to the free fall problem.

The preceding observations motivate the following

> CONJECTURE: If $x(t) = x_0 + f(t - t_0)$ describes the general solution of a differential equation of the form $\ddot{x} = F(x)$ then necessarily $F(x) = \text{constant}$.

—the proof of which is, in fact, easy. The conjecture identifies a seldom-remarked **uniqueness property of the free fall problem**, the effects of which will haunt this work.

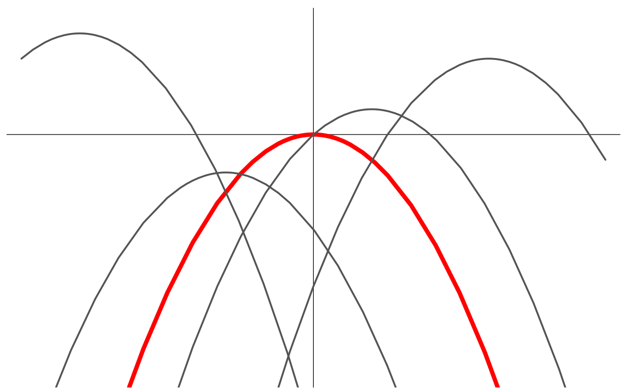

FIGURE 1: *Shown in red is a graph of the "primitive" free fall solution* (5.1). *Other solutions of* $\ddot{x} + g = 0$ *are seen to have the same shape; i.e., to be translationally equivalent.*

4. Lagrangian, Hamiltonian and 2-point action. If, as is most natural, we take the free fall Lagrangian to be given by

$$L(\dot{x}, x) \equiv \tfrac{1}{2}m\dot{x}^2 - U(x) \tag{7.1}$$

$$U(x) = mgx \tag{7.2}$$

then

$$p \equiv \partial L/\partial\dot{x} = m\dot{x}$$

and the Hamiltonian becomes

$$H(p,x) = p\dot{x} - L(\dot{x},x) = \tfrac{1}{2m}p^2 + mg\,x \tag{8}$$

The *action* associated with any test path $x(t)$ is given by

$$S[x(t)] \equiv \int_{t_0}^{t_1} L\big(\dot{x}(t),x(t)\big)\,dt$$

Take $x(t)$ to be in fact a *solution* of the equation of motion. Take it, more particularly, to be the solution described at (2). Then

$$\begin{aligned} S[a+bt-\tfrac{1}{2}gt^2] &= \int_{t_0}^{t_1} L(b-gt, a+bt-\tfrac{1}{2}gt^2)\,dt \\ &= m\Big\{(\tfrac{1}{2}b^2 - ag)(t_1-t_0) - bg(t_1^2-t_0^2) + \tfrac{1}{3}g^2(t_1^3-t_0^3)\Big\} \end{aligned}$$

Take a and b to be given by (3). Then *Mathematica* supplies[3]

$$S(x_1,t_1;x_0,t_0) = \tfrac{1}{2}m\left\{\frac{(x_1-x_0)^2}{t_1-t_0} - g(x_0+x_1)(t_1-t_0) - \tfrac{1}{12}g^2(t_1-t_0)^3\right\} \tag{9}$$

This is the action functional associated with (not must some arbitrary "test path" linking $(x_0,t_0) \longrightarrow (x_1,t_1)$ but with) the *dynamical path* linking those spacetime points: it is not a functional but a *function* of the parameters $(x_1,t_1;x_0,t_0)$ that serve by (4) to define the path, and is, for reasons now obvious, called the "2-point action function."

It becomes notationally sometimes convenient to drop the $_1$'s from x_1 and t_1, and to write $S(x,t;x_0,t_0)$. We observe that the free fall action function (9) gives back the more frequently encountered free particle action

$$\begin{aligned} &\downarrow \\ &= \tfrac{1}{2}m\frac{(x_1-x_0)^2}{t_1-t_0} \quad \text{when the gravitational field is turned off: } g\downarrow 0 \end{aligned}$$

5. Energy & momentum. The total energy of our freely falling particle is given by

$$\begin{aligned} E(t) &= \tfrac{1}{2}m\dot{x}^2(t) + mg\,x(t) \\ &= \tfrac{1}{2m}p^2(t) + mg\,x(t) \\ &= \text{instantaneous numerical value of the Hamiltonian} \end{aligned} \tag{10}$$

Taking $x(t)$ to be given by (2) we compute

$$E(t) = \tfrac{1}{2}m(b^2 + 2ga) = \text{constant} \tag{11}$$

[3] For the first occurance of this result in my own writing, see QUANTUM MECHANICS (1967), Chapter 1, page 21. But beware: one must reverse the sign of g to achieve agreement with results quoted there.

Energy conservation is no surprise: it follows directly from the t-independence of the Hamiltonian (which is to say: from the t-independence of the potential).

Working from (2) we find that the momentum

$$p(t) = m(b - gt) \tag{12}$$

which follows also from

$$\begin{aligned} p(t) &= \sqrt{2m[E - mg\,x(t)]} \\ &= \sqrt{m^2(b^2 + 2ga) - 2m^2 g(a + bt - \tfrac{1}{2}gt^2)} \\ &= \sqrt{m^2(b^2 - 2gbt + g^2t^2)} \\ &= \sqrt{m^2(b - gt)^2} \end{aligned}$$

Working from (4)—or (more simply) by drawing upon (3)—we obtain

$$p(t) = m\Big\{\frac{x_1 - x_0}{t_1 - t_0} + \tfrac{1}{2}g(t_0 - 2t + t_1)\Big\} \tag{13}$$

which means that a particle that moves along the trajectory $(x_0, t_0) \longrightarrow (x_1, t_1)$ must be launched with momentum

$$p(t_0) = m\Big\{\frac{x_1 - x_0}{t_1 - t_0} + \tfrac{1}{2}g\,(t_1 - t_0)\Big\}$$

and arrives with momentum

$$p(t_1) = m\Big\{\frac{x_1 - x_0}{t_1 - t_0} - \tfrac{1}{2}g\,(t_1 - t_0)\Big\}$$

General theory asserts, and computation confirms, that these equations could also have been obtained from

$$\left.\begin{aligned} p(t_1) &= +\frac{\partial S(x_1, t_1; x_0, t_0)}{\partial x_1} \\ p(t_0) &= -\frac{\partial S(x_1, t_1; x_0, t_0)}{\partial x_0} \end{aligned}\right\} \tag{14}$$

Notice that

$$\Delta p \equiv p(t_1) - p(t_0) = -mg\,(t_1 - t_0) = \int_{t_0}^{t_1} (-mg)\,dt = \text{impulse}$$

and that, according to (13), $p(t)$ interpolates *linearly* between its initial and final values.

6. Noether's theorem. It is to a mathematician (Emmy Noether, 1918) that physicists owe recognition of an illuminating connection between

- an important class of conservation laws and
- certain invariance properties ("symmetries") of the dynamical action.

We look here only to some particular instances of that connection, as they relate specifically to the free fall problem.

We saw in §3 that **time translation** generally, and infinitesimal time translation $(x,t) \longmapsto (x, t+\delta t)$ more particularly,[4] maps solutions to other solutions of the free fall equations:

$$(x-x_0)+\tfrac{1}{2}g(t-t_0)^2=0 \quad\longmapsto\quad (x-x_0)+\tfrac{1}{2}g(t-(t_0-\delta t))^2=0$$

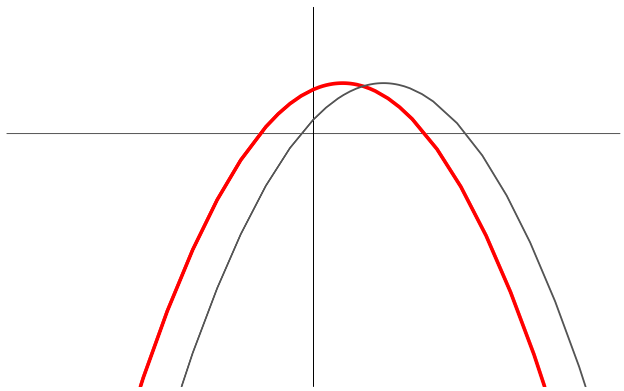

FIGURE 2: *Graphs of a free fall and of its temporal translate. If the initial motion is identified by specification of its endpoints then to identify its translate one must modify* both *endpoints:*

$$(x_1,t_1;x_0,t_0)\longmapsto(x_1,t_1+\delta t;x_0,t_0+\delta t)$$

Noether argues[5] that

$$\begin{aligned}\delta S &= S(x_1,t_1+\delta t;x_0,t_0+\delta t)-S(x_1,t_1;x_0,t_0)\\ &= J(\dot x,x)\Big|_{t_0}^{t_1}\cdot\delta t \quad\text{with}\quad J(\dot x,x)\equiv-\Big[p(\dot x,x)\dot x-L(\dot x,x)\Big]\end{aligned}$$

But it is a manifest implication of (9) that

$$\delta S=0\quad:\quad\text{the dynamnical action is } t\text{-translation } \textit{invariant}$$

and from this it follows that

$$\Big[p(\dot x,x)\dot x-L(\dot x,x)\Big]_{t_1}=\Big[p(\dot x,x)\dot x-L(\dot x,x)\Big]_{t_0}$$

which asserts simply that *energy is conserved*.

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[4] Noether worked within the framework provided by the calculus of variations, so found it natural to assign special importance to *infinitesimal* transformations.

[5] See CLASSICAL MECHANICS (1983), page 163.

Similarly . . . we saw in §3 that **space translation** generally, and infinitesimal space translation $(x,t)\longmapsto(x+\delta x,t)$ more particularly, maps solutions to other solutions of the free fall equations:

$$(x-x_0)+\tfrac{1}{2}g(t-t_0)^2=0 \quad\longmapsto\quad (x-(x_0-\delta x))+\tfrac{1}{2}g(t-t_0)^2=0$$

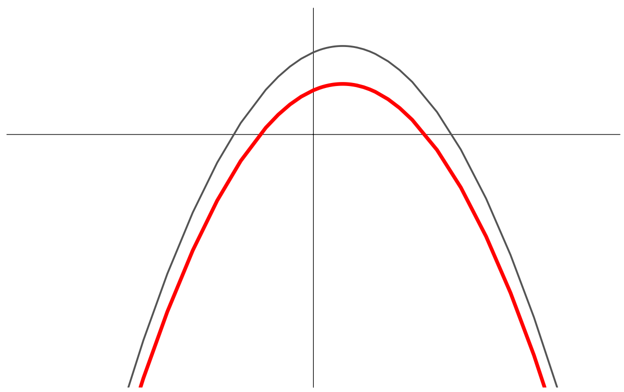

FIGURE 3: *Graphs of a free fall and of its spatial translate. Again, if we adopt endpoint specification then we must write*

$$(x_1,t_1;x_0,t_0)\longmapsto(x_1+\delta x,t_1;x_0+\delta x,t_0)$$

Noether's argument in this instance supplies

$$\begin{aligned}\delta S&=S(x_1,t_1+\delta t;x_0,t_0+\delta t)-S(x_1,t_1;x_0,t_0)\\&=J(\dot x,x)\Big|_{t_0}^{t_1}\cdot\delta t\quad\text{where now}\quad J(\dot x,x)\equiv p(\dot x,x)\end{aligned}$$

But it is a manifest implication of (9) that—owing to the presence of the (x_0+x_1)-term (which vanishes in the free particle limit $g\downarrow 0$)—

$$\delta S\neq 0\quad:\quad\text{the dynamnical action is }\underline{\text{not }x\text{-translation invariant}}$$

and from this it follows that

$$\Big[p(\dot x,x)\Big]_{t_1}\neq\Big[p(\dot x,x)\Big]_{t_0}\quad:\quad\text{momentum is }\textit{not conserved}$$

Evidently it is not the "symmetry of the space of motions" that matters: it is *symmetry of the dynamical* <u>*action*</u> that gives rise to conservation laws.

7. 2-point Hamilton-Jacobi equations. Classical mechanics, pursued to its depths, supplies the information that the dynamical action function satisfies a pair of (generally non-linear) partial differential equations—namely, the

Hamilton-Jacobi equation

$$H\left(\tfrac{\partial S(x_1,t_1;\bullet,\bullet)}{\partial x_1}, x_1\right) + \tfrac{\partial S(x_1,t_1;\bullet,\bullet)}{\partial t_1} = 0$$

and its time-reversed companion

$$H\left(\tfrac{\partial S(\bullet,\bullet;x_0,t_0)}{\partial x_0}, x_0\right) - \tfrac{\partial S(\bullet,\bullet;x_0,t_0)}{\partial t_0} = 0$$

In the present context those equations read

$$\tfrac{1}{2m}\left(\tfrac{\partial S(x_1,t_1;\bullet,\bullet)}{\partial x_1}\right)^2 + mgx_1 + \tfrac{\partial S(x_1,t_1;\bullet,\bullet)}{\partial t_1} = 0 \tag{15.1}$$

$$\tfrac{1}{2m}\left(\tfrac{\partial S(\bullet,\bullet;x_0,t_0)}{\partial x_0}\right)^2 + mgx_0 - \tfrac{\partial S(\bullet,\bullet;x_0,t_0)}{\partial t_0} = 0 \tag{15.2}$$

and *Mathematica* confirms that the $S(x_1, t_1; x_0, t_0)$ of (9) does in fact satisfy those equations.

There are, of course, infinitely many *other* bi-functions $F(x_1, t_1; x_0, t_0)$ that satisfy the H-J system (15). We have yet to describe the sense in which S occupies a distinguished place within that population.

8. Separated solutions of the 1-point Hamilton-Jacobi equation. In some respects deeper, and in all respects simpler and more transparent . . . than the theory of 2-point H-J functions is the theory of 1-point H-J functions, central to which is the (solitary) Hamilton-Jacobi equation

$$H\left(\tfrac{\partial S(x,t)}{\partial x}, x\right) + \tfrac{\partial S(x,t)}{\partial x} = 0 \tag{16}$$

As—briefly—to the meaning of that equation: given a function $S(x) \equiv S(x, 0)$ the fundamental relation

$$p(x) = \tfrac{\partial S(x,0)}{\partial x} \tag{17}$$

serves to describe a *curve* $\mathcal{C}_0$ on phase space (*i.e.*, on (x, p)-space). The H-J equation (16) describes the *dynamical motion of that curve*:

$$\mathcal{C}_0 \xrightarrow[\text{Hamiltonian-induced phase flow}]{} \mathcal{C}_t \tag{18}$$

We will have occasion later to describe concrete instances of (18).

In the case of interest (16) reads

$$\tfrac{1}{2m}\left(\tfrac{\partial S(x,t)}{\partial x}\right)^2 + mgx + \tfrac{\partial S(x,t)}{\partial t} = 0 \tag{19}$$

What happens if we attempt to solve that non-linear partial differential equation by separation of variables? Write

$$S(x, t) = W(x) + F(t)$$

and obtain

$$\tfrac{1}{2m}\left(\tfrac{\partial W(x)}{\partial x}\right)^2 + mgx = -\tfrac{\partial F(t)}{\partial t}$$

whence

$$\left.\begin{aligned} \tfrac{1}{2m}\left(\tfrac{dW(x)}{dx}\right)^2 + mgx &= +E \\ \tfrac{dF(t)}{dt} &= -E \end{aligned}\right\} \quad : \quad E \text{ is a separation constant}$$

Immediately

$$\begin{aligned} F(t) &= F_0 - Et \\ W(x) &= \int^x \sqrt{2m[E - mg\xi]}\, d\xi \\ &= \tfrac{2}{3mg}\sqrt{2m[E - mgx]^3} + W_0 \end{aligned}$$

which supply this E-parameterized family of particular H-J functions

$$S(x,t;E) = \tfrac{2}{3mg}\sqrt{2m[E - mgx]^3} - Et + S_0 \tag{20}$$

Here we have lumped the additive constants: $S_0 \equiv W_0 + F_0$.[6]

We will revisit this topic after we have acquired quantum mechanical reason to do so: for the moment I must be content mere to set the stage. We are in the habit of thinking that *solutions obtained by separation can be combined to produce the general solution* of a partial differential equation. How is that to be accomplished in the present instance? Quantum theory will motivate us to ask this sharper question: How can the H-J functions $S(x,t;E)$ be combined to produce the $S(x_1,t_1;x_0,t_0)$ of (9)?[7]

9. Hamiltonian methods. The free fall Hamiltonian

$$H(p,x) = \tfrac{1}{2m}p^2 + mgx$$

supplies canonical equations

$$\left.\begin{aligned} \dot{x} &= +\partial H/\partial p = \tfrac{1}{m}p \\ \dot{p} &= -\partial H/\partial x = -mg \end{aligned}\right\} \tag{21}$$

which are transparently equivalent to (1). The motion of an arbitrary observable $A(p,x,t)$ is therefore given by

$$\begin{aligned} \dot{A} &= \frac{\partial A}{\partial x}\frac{\partial H}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial H}{\partial x} + \frac{\partial A}{\partial t} \equiv [A, H\,] + \frac{\partial A}{\partial t} \\ &= \tfrac{1}{m}pA_x - mgA_p + A_t \end{aligned} \tag{22}$$

Energy conservation is, from this point of view, immediate

$$\dot{H} = [H, H\,] = 0$$

and so is momentum non-conservation:

$$\dot{p} = [\,p, H\,] = [\,p, mgx] = -mg \neq 0$$

[6] Because it is always useful to inquire "What happens in the free particle limit to the gravitational result in hand?" we expand in powers of g and obtain the curious result

$$S = \left\{S_0 - \tfrac{4}{3}\tfrac{E^2}{P}g^{-1}\right\} + \left\{Px - Et\right\} - \tfrac{1}{2}\tfrac{m^2x^2}{P}g + \cdots$$

where $P \equiv \sqrt{2mE}$.

[7] General theory relating to those questions is developed in Appendix B: "Legendre transformation to/from the 'energy representation' and its Fourier-analytic quantum analog" to the class notes cited in footnote 5.

Phase flow carries

$$(x,p)_0 \xrightarrow[\text{phase flow}]{} (x,p)_t = (x_0 + \tfrac{1}{m}p_0 t - \tfrac{1}{2}gt^2,\; p_0 - mgt) \tag{23}$$

Necessarily $(x,p)_0$ and $(x,p)_t$ lie on the same isoenergetic curve, and those E-parameterized curves

$$\tfrac{1}{2m}p^2 + mgx = E$$

are parabolic, oriented as shown in the following figure:

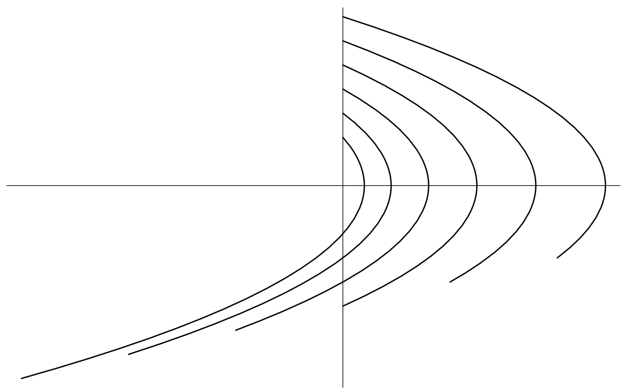

FIGURE 4: *Effect of phase flow on a representative population of phase points, computed on the basis of* (23). *The x-axis runs $\rightarrow$, the p-axis runs $\uparrow$. I have set $m = g = 1$ and assigned to the phase points the initial coordinates* $(0, 0.4)$, $(0, 0.6)$, $(0, 0.8)$, $(0, 1.0)$, $(0, 1.2)$ *and* $(0, 1.4)$. *The coaxial parabolas all have the same shape, and are conveniently distinguished/identified by their x-intercepts*: $x_E = E/mg$.

We now adjust our viewpoint, agreeing to look upon . . .

10. Gravitation as an artifact of non-inertiality. The root idea is that when we encounter an equation of motion of the form

$$\boldsymbol{F} + m\boldsymbol{G} = m\ddot{\boldsymbol{x}}$$

we should think of the $m\boldsymbol{G}$-term not as a "force that happens to adjust its strength in proportion to the mass of the particle upon which it acts" (shades of Galileo!)—indeed, not as a "force" at all—but as an "acceleration term" that has slipped to the wrong side of the equality: that we should instead write

$$\boldsymbol{F} = m(\ddot{\boldsymbol{x}} - \boldsymbol{G})$$

and interpret the non-Newtonian structure of the expression on the right to signal that $\boldsymbol{x}$ must refer to a *non-inertial coordinate system*. To adopt such a view[8] is to dismiss gravitation as a "fictitious force," akin to the centrifugal

[8] Hard to do if, as I have, you've every been hospitalized by a fall, though we speak here of nothing less than Einstein's lofty Equivalence Principle.

and Coriolis "forces." The idea can be implemented, in a degree of generality sufficient for the purposes at hand,[9] as follows:

Let x and x, which refer to Cartesian coordinatizations of 1-space, stand in the simple relation

$$x = x + a(t) \tag{24}$$

Evidently $a(t)$ describes the instantaneous location, relative to the $\mathcal{X}$-frame, of the $\mathcal{X}$-origin (conversely, $-a(t)$ describes the instantaneous location, relative to the $\mathcal{X}$-frame, of the $\mathcal{X}$-origin):

$$\begin{aligned} x(0,t) &= +a(t) \\ x(0,t) &= -a(t) \end{aligned}$$

<u>Assume $\mathcal{X}$ to be inertial</u>: assume, in other words, that the force-free motion of a mass m relative to $\mathcal{X}$ can be described

$$m\ddot{x} = 0 \tag{25.1}$$

In x-coordinates that statement becomes

$$m(\ddot{x} + \ddot{a}) = 0 \tag{25.2}$$

To recover (1) we have only to set $a(t) = \frac{1}{2}gt^2$. (We might, more generally, set $a(t) = a_0 + a_1 t + \frac{1}{2}gt^2$ but in the interest of simplicity I won't.) Then

$$\begin{aligned} x &= x + \tfrac{1}{2}gt^2 \\ x &= x - \tfrac{1}{2}gt^2 \end{aligned}$$

Motion which is seen to be free with respect to $\mathcal{X}$ is seen as free fall to the left when referred to the $\mathcal{X}$-frame which $\mathcal{X}$ sees to be accelerating to the right.

Turning now from the kinematic to the Lagrangian dynamical aspects of the problem, and taking

$$L(\dot{x}, x) = \tfrac{1}{2}m\dot{x}^2 \quad : \quad \text{FREE PARTICLE LAGRANGIAN}$$

as an obvious point of departure, we construct

$$L(\dot{x}, x, t) \equiv L(\dot{x} + gt, x + \tfrac{1}{2}gt^2) = \tfrac{1}{2}m(\dot{x} + gt)^2$$

which (gratifyingly, but not at all to our surprise) gives back (1):

$$\tfrac{d}{dt}m(\dot{x} + gt) = m(\ddot{x} + g) = 0$$

The Lagrangian $L(\dot{x}, x, t)$ does not much resemble the Lagrangian encountered at (7), but in fact they *differ only by a gauge term*:

$$\big[\tfrac{1}{2}m(\dot{x} + gt)^2\big] = \big[\tfrac{1}{2}m\dot{x}^2 - mgx\big] + \tfrac{d}{dt}\big(mgxt + \tfrac{1}{6}mg^2t^3\big)$$

[9] For a fairly detailed general account of the theory of fictitious forces see pages 101–110 in the class notes cited several times previously.[5]

Our program—the objective of which is to recover all the unfamiliar details of free fall physics from the simpler details of free particle physics—proceeds from the elementary device of stepping from an inertial frame to a uniformly accelerated frame, but is seen now to involve rather more than a simple change of coordinates $x \longrightarrow {\color{red}x} = x - \frac{1}{2}gt^2$: it is a 2-STEP PROCESS

$$\left.\begin{aligned} x &\longrightarrow {\color{red}x} = x - \tfrac{1}{2}gt^2 \\ L &\longrightarrow {\color{red}L} = L + \tfrac{d}{dt}\Lambda \end{aligned}\right\} \tag{26}$$

involving a coordinate transformation *and a synchronized gauge transformation*. We must attend carefully to conceptual/notational distinctions at risk of becoming confused (or—which is almost as bad—confusing). I spell out the detailed meaning and some of the immediate implications of the second of the preceding statements:

$$\begin{aligned} L(\dot{x}) &\equiv \tfrac{1}{2}m\dot{x}^2 \\ &\downarrow \\ {\color{red}L}({\color{red}\dot{x}},{\color{red}x}) &\equiv \tfrac{1}{2}m{\color{red}\dot{x}}^2 - mg{\color{red}x} \\ &= L({\color{red}\dot{x}} + gt) + \tfrac{d}{dt}\Lambda({\color{red}x},t) \\ &\qquad\qquad \Lambda({\color{red}x},t) \equiv -mg{\color{red}x}t - \tfrac{1}{6}mg^2t^3 \\ &= \tfrac{1}{2}m({\color{red}\dot{x}} + gt)^2 - mg\big[{\color{red}\dot{x}}t + {\color{red}x} + \tfrac{1}{2}gt^2\big] \end{aligned} \tag{27}$$

Collaterally

$$\begin{aligned} p &\equiv \partial L/\partial\dot{x} = m\dot{x} \\ &\downarrow \\ {\color{red}p} &\equiv \partial{\color{red}L}/\partial{\color{red}\dot{x}} = m{\color{red}\dot{x}} \\ &= (\partial L/\partial\dot{x})\underbrace{(\partial\dot{x}/\partial{\color{red}\dot{x}})}_{=\,(\partial x/\partial{\color{red}x})\,=\,1} + (\partial\Lambda/\partial{\color{red}x}) \\ &= p - mgt \end{aligned}$$

Look now to the transformation of the Hamiltonian. Generally, the fact that the Lagrangian responds as a scalar to coordinate transformations implies that the Hamiltonian does too: that would give

$$\begin{aligned} H(p,x) &\equiv p\dot{x} - L = \tfrac{1}{2m}p^2 \\ &\downarrow \\ {\color{red}H}({\color{red}p},{\color{red}x}) &\equiv {\color{red}p}{\color{red}\dot{x}} - \tfrac{1}{2}m({\color{red}\dot{x}} + gt)^2\Big|_{{\color{red}\dot{x}} \to ({\color{red}p} - mgt)/m} \\ &= \tfrac{1}{2m}{\color{red}p}^2 - gt{\color{red}p} \\ &= \tfrac{1}{2m}({\color{red}p} - mgt)^2 - \tfrac{1}{2}mg^2t^2 \end{aligned} \tag{28.1}$$

But gauge adjustment of the Lagrangian is found[10] to alter the *definition* of p (specifically, $p = m(\dot{x} + gt)$ becomes $p = m(\dot{x} + gt) + \partial\Lambda/\partial x = m\dot{x}$, which is to say: $p_{\text{old}} - mgt = p_{\text{new}}$) and to cause the Hamiltonian to pick up a subtractive t-partial of the gauge function. The net effect can in the present instance be described

$$\begin{aligned} &\downarrow \\ &= \tfrac{1}{2m}p^2 - \tfrac{1}{2}mg^2t^2 - \tfrac{\partial}{\partial t}\Big[-mgxt - \tfrac{1}{6}mg^2t^3\Big] \\ &= \tfrac{1}{2m}p^2 + mgx \end{aligned} \tag{28.2}$$

We will soon be motivated to look more closely to the line of argument just sketched.

11. Contact with the theory of canonical transformations. It is important to bear in mind that the question "To gauge or not to gauge?" is physically inconsequential: it is resolved by whim of the physicist, in response to formal (or perhaps merely æsthetic) considerations. In the following discussion I pursue both options. Later it will emerge that one is more readily adapted to quantum mechanics than the other.

UNGAUGED FORMALISM Underlying all is the ungauged statement

$$L(\dot{x}, x, t) \equiv L(\dot{x} + gt, x + \tfrac{1}{2}gt^2) = \tfrac{1}{2}m(\dot{x} + gt)^2$$

from which follow $p = m(\dot{x} + gt) = m\dot{x} = p$ and the Hamiltonian (28.1). From

$$\left.\begin{aligned} x &\longrightarrow x = x - \tfrac{1}{2}gt^2 \\ p &\longrightarrow p = p \end{aligned}\right\} \tag{29}$$

(where the first line describes a t-dependent "point transformation" and the two lines taken together describe the induced "extended point transformation") it follows trivially that

$$[x, p] \equiv \frac{\partial x}{\partial x}\frac{\partial p}{\partial p} - \frac{\partial p}{\partial x}\frac{\partial x}{\partial p} = 1$$

which is to say: the transformation (29) is *canonical*. At any given t the equations (29) describe a *rigid translation of the phase plane*—a translation that, as it happens, leaves invariant the population of phase curves shown in Figure 4.

Classical mechanics supplies two distinct techniques for "generating" canonical transformations—one associated with the name of Legendre, the other with that of Lie. We look first to the former:[11]

Legendre recognizes generators of four standard types

$$F_1(x, x), \quad F_2(p, x), \quad F_3(x, p), \quad F_4(p, p)$$

[10] See page 198 in CLASSICAL MECHANICS (1983).
[11] See pages 224–234 in CLASSICAL MECHANICS (1983).

—each of which is a function of "half the old phase coordinates (x, p) and half the new (x, p)." A moment's tinkering leads us to the Type 2 generator

$$F_2(p, x) \equiv (x - \tfrac{1}{2}gt^2)p \tag{30}$$

from which we recover (29) by writing

$$\left.\begin{aligned} p &= \partial F_2/\partial x = p \\ x &= \partial F_2/\partial p = x - \tfrac{1}{2}gt^2 \end{aligned}\right\} \tag{31}$$

Notice that in the limit $g \downarrow 0$ we are left with a well-known "generator of the identity": $F_2(p, x) = xp$. Notice also that if we write

$$H(p, x) = \tfrac{1}{2m}p^2 \quad \text{and} \quad H(p, x) = \tfrac{1}{2m}p^2 - gtp$$

then[12]

$$= H\big(p(p, x), x(p, x)\big) + \partial F_2/\partial t \tag{32}$$

The "extra term" $\partial F_2/\partial t = -gtp$ is brought into play by the circumstance that the transformation (29) is t-dependent: only when such terms are absent can one say that the Hamiltonian transforms as a scalar.

Lie would have us undertake to achieve (29) not "all at once" (as Legendre did) but incrementally, by iteration of *infinitesimal* canonical transformations. His idea can be implemented as follows: in place of (29) write

$$\left.\begin{aligned} x(u) &= x - \tfrac{1}{2}gt^2u \\ p(u) &= p \end{aligned}\right\} \tag{33}$$

Here u is a dimensionless parameter that in effect "tunes the strength" of the gravitational field and enables us to *smoothly interpolate* between free particle physics and free fall physics: more particularly

$$\begin{aligned} x(0) &= x \\ p(0) &= p \end{aligned} \quad \text{and} \quad \begin{aligned} x(1) &= x \\ p(1) &= p \end{aligned}$$

It follows from (33) that

$$\begin{aligned} \tfrac{d}{du}x(u) &= -\tfrac{1}{2}gt^2 \\ \tfrac{d}{du}p(u) &= 0 \end{aligned}$$

These equations can be made to assume the design of Hamilton's canonical equations of motion

$$\left.\begin{aligned} \tfrac{d}{du}x(u) &= +\tfrac{\partial}{\partial p}G(p, x) = -[G, x] \\ \tfrac{d}{du}p(u) &= -\tfrac{\partial}{\partial x}G(p, x) = -[G, p] \end{aligned}\right\} \tag{34}$$

[12] See pages 234–239 in CLASSICAL MECHANICS (1983).

provided we set

$$G(p,x) = -\tfrac{1}{2}gt^2p \tag{35}$$

Formal iteration leads to this description of the *solution* of (34):

$$\begin{aligned}
x(u) &= x - u[G,x] + \tfrac{1}{2!}u^2[G,[G,x]] - \tfrac{1}{3!}u^3[G,[G,[G,x]]] + \cdots\\
&= x - u\tfrac{1}{2}gt^2\\
p(u) &= p - u[G,p] + \tfrac{1}{2!}u^2[G,[G,p]] - \tfrac{1}{3!}u^3[G,[G,[G,p]]] + \cdots\\
&= p
\end{aligned}$$

from which we recover (29) at $u = 1$. It is the fact that x and p enter *linearly* into the design of $G(p,x)$ that accounts for the exceptional simplicity of these results (*i.e.*, for the disappearance of terms of $O(u^2)$).

Notice that if $H(p,x) = \frac{1}{2m}p^2$ then

$$H - u[G,H] + \tfrac{1}{2!}u^2[G,[G,H]] - \tfrac{1}{3!}u^3[G,[G,[G,H]]] + \cdots = H$$

which is to say: Lie's iterative process does *not* in this instance lead to the correct transformed Hamiltonian. It leads to $H(p,x) = \frac{1}{2m}p^2$, to which one must "by hand" add the "extra term" $\partial F_2/\partial t = -gtp$.

Hamilton-Jacobi theory (see again §7) serves to describe the *relation between* the Lie and Legendre generators of any given canonical transformation. Looking now to the particulars of the case at hand ... the transformation (29) can be recovered from the following $(u-u)$-parameterized *family* of Legendre generators:[13]

$$F(p,u;x,u) \equiv [x - \tfrac{1}{2}g(u-u)t^2]p \tag{36}$$

We compute

$$\begin{aligned}
G\big(p, \tfrac{\partial F(p,u;x,u)}{\partial p}\big) &= -\tfrac{1}{2}gt^2p\\
\tfrac{\partial F(p,u;x,u)}{\partial u} &= -\tfrac{1}{2}gt^2p
\end{aligned}$$

$$\begin{aligned}
G\big(\tfrac{\partial F(p,u;x,u)}{\partial x}, x\big) &= -\tfrac{1}{2}gt^2p\\
\tfrac{\partial F(p,u;x,u)}{\partial u} &= +\tfrac{1}{2}gt^2p
\end{aligned}$$

from which follow this pair of "2-point Hamilton-Jacobi equations:"

$$\left.\begin{aligned}
G\big(p, \tfrac{\partial F(p,u;x,u)}{\partial p}\big) - \tfrac{\partial F(p,u;x,u)}{\partial u} &= 0\\
G\big(\tfrac{\partial F(p,u;x,u)}{\partial x}, x\big) + \tfrac{\partial F(p,u;x,u)}{\partial u} &= 0
\end{aligned}\right\} \tag{37}$$

The signs are a bit funny because F is a Type 2 generator, while the S of §7 is a generator of Type 1.

[13] Here I write $u \to u - u$ to introduce a "floating initial value of the evolution parameter"—this so that both $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial u}$ will be meaningful.

GAUGED FORMALISM We build upon—meaning here literally "on top of"—the theory just developed. Specifically, we write

$$
\begin{aligned}
L(\dot{x},x,t) &= \tfrac{1}{2}m(\dot{x}+gt)^2\\
&\downarrow\\
L(\dot{x},x,t) &\equiv L(\dot{x},x,t)+\tfrac{d}{dt}\Lambda(x,t)\\
&\qquad\qquad\qquad \Lambda(x,t)\equiv -mgxt-\tfrac{1}{6}mg^2t^3\\
&= \tfrac{1}{2}m\dot{x}^2-mgx
\end{aligned}
$$

to describe the gauge transformation that brings L to *standard* free fall form. We write

$$
\left.\begin{aligned}
x &\longrightarrow x = x\\
p &\longrightarrow p = p-mgt
\end{aligned}\right\} \tag{38}
$$

to describe the induced canonical recoordinatization of phase space. The gauged Lagrangian can in this notation be described

$$
L(\dot{x},x)=\tfrac{1}{2}m\dot{x}^2-mgx
$$

according to which we expect to be able to write

$$
p=m\dot{x}\qquad\text{and}\qquad H(p,x)=\tfrac{1}{2m}p^2+mgx
$$

And indeed: $m\dot{x}=m\dot{x}=m\frac{\partial}{\partial p}\big\{\frac{1}{2m}p^2-gtp\big\}=p-mgt=p$ while a general argument

$$
\begin{aligned}
H(p,x)=\dot{x}p-L &= \dot{x}(p+\tfrac{\partial}{\partial x}\Lambda)-(L+\dot{x}\tfrac{\partial}{\partial x}\Lambda+\tfrac{\partial}{\partial t}\Lambda)\\
&= H(p,x)-\tfrac{\partial}{\partial t}\Lambda(x,t)\\
&= H(p-\tfrac{\partial}{\partial x}\Lambda,x)-\tfrac{\partial}{\partial t}\Lambda(x,t)
\end{aligned} \tag{39}
$$

can be used in the present instance to supply

$$
\begin{aligned}
H(p,x) &= \big[\tfrac{1}{2m}(p+mgt)^2-gt(p+mgt)\big]-\big[-mgx-\tfrac{1}{2}mg^2t^2\big]\\
&= \tfrac{1}{2m}p^2+mgx
\end{aligned}
$$

—precisely as was anticipated.

The canonical transformation (38) is readily seen to be Legendre-generated by

$$
F_2(p,x)\equiv x(p+mgt)
$$

and Lie-generated by

$$
G(p,x)=mgtx
$$

If we construct

$$
F(p,u;x,u)\equiv x[p+(u-u)mgt]
$$

we confirm by quick calculation that the Lie/Legendre generators stand in the Hamilton-Jacobi relationship

$$\left.\begin{aligned} G\big(p, \tfrac{\partial F(p,u;x,u)}{\partial p}\big) - \tfrac{\partial F(p,u;x,u)}{\partial u} &= 0 \\ G\big(\tfrac{\partial F(p,u;x,u)}{\partial x}, x\big) + \tfrac{\partial F(p,u;x,u)}{\partial u} &= 0 \end{aligned}\right\} \tag{40}$$

The Hamiltonian $H(p,x)$ acquired an additive "extra term" by the gauge mechanism (39). But if we had been *unaware* that the canonical transformation (38) was recommended to our attention by a gauged Lagrangian then we would not have know to make such an adjustment, but on the other hand would have been led (by general theory and the circumstance that (39) is t-dependent) to introduce the additive term $\partial F_2/\partial t$. It is important to recognize that these two procedures come to essentially the same thing

$$\begin{aligned} \tfrac{\partial}{\partial t} F_2(p,x) &= mgx \\ -\tfrac{\partial}{\partial t}\Lambda(x,t) &= mgx + \tfrac{1}{2}mg^2t^2 \end{aligned}$$

and would come to *exactly* the same thing if, in place of $F_2(p,x) \equiv x(p+mgt)$, we agreed to write $F_2(p,x) \equiv x(p+mgt) + \frac{1}{6}mg^2t^3$. This we can do with impunity, since in all applications the new F_2 precisely mimics the old one. It would, however, be a mistake to make *both* additive adjustments. It was mainly to clarify this point that I have allowed myself to schlog so pedantically through the preceding material.

Bringing (29) to (38) we obtain the conflated canonical transformation

$$\left.\begin{aligned} x &= x - \tfrac{1}{2}gt^2 \\ p &= p - mgt \end{aligned}\right\} \tag{41}$$

of which the Legendre generators have the form

$$F_2(p,x) \equiv (x - \tfrac{1}{2}gt^2)(p+mgt) + f(t) \quad : \quad f(t) \text{ arbitrary}$$

The transform of the free particle Hamiltonian $H(p,x) = \frac{1}{2m}p^2$ is

$$\begin{aligned} H(p,x) &= \tfrac{1}{2m}(p+mgt)^2 + \big[-gtp + mg(x + \tfrac{1}{2}gt^2) - \tfrac{3}{2}mg^2t^2 + f\,'(t)\big] \\ &= \tfrac{1}{2m}p^2 + mgx + \underbrace{\big[f\,'(t) - \tfrac{1}{2}mg^2t^2\big]}_{\text{vanishes if we set } f(t) = \frac{1}{6}mg^2t^3} \end{aligned}$$

The Lie generator is

$$G(p,x) \equiv mgtx - \tfrac{1}{2}gt^2p$$

If we define

$$F(p,u;x,u) \equiv [x - (u-u)\tfrac{1}{2}gt^2][p + (u-u)mgt] + \tfrac{1}{4}mg^2t^3(u-u)^2$$

then we find

$$\begin{aligned} G\big(p, \tfrac{\partial F(p,u;x,u)}{\partial p}\big) - \tfrac{\partial F(p,u;x,u)}{\partial u} &= 0 \\ G\big(\tfrac{\partial F(p,u;x,u)}{\partial x}, x\big) + \tfrac{\partial F(p,u;x,u)}{\partial u} &= 0 \end{aligned}$$

The transformations described above were designed to extract the physics of free fall from the physics of motion in the absence of forces. But they are readily generalized, readily inverted: by straightforward modification they can be used to adjust the value of g, and can in particular be used to turn g off—thus to reverse the trend of the argument, to recover free motion from free fall.

The discussion acquires incidental interest from the fact that it provides concrete illustrations of many of the most characteristic *general principles of Hamiltonian mechanics*—principles that in the next section we apply to a different objective.

12. Galilean covariance. Suppose now that at (24) we had assumed $a(t)$ to depended *linearly* on t:

$$x = x + vt \tag{42}$$

Then (compare (25))

$$m\ddot{x} = 0 \qquad \text{becomes} \qquad m\ddot{x} = 0 \tag{43}$$

If, on the other hand, (43) is *stipulated* then (42) is in effect *forced*: we have come upon the birthplace of Galilean relativity, the source of the notion that inertial frames move uniformly with respect to one another, the origin of the statement that "free motion is a Galilean covariant concept." More to the immediate point: if x and x stand in the relation (42) then

$$m\ddot{x} = -mg \qquad \text{becomes} \qquad m\ddot{x} = -mg \tag{44}$$

which gives back (43) in the special case $g = 0$ and informs us that "free *fall* is a Galilean covariant concept." We look now to some of the implications of that fact.

Look to the solution $x(t) = a + bt - \frac{1}{2}gt^2$ of $m\ddot{x} = -mg$. When referred to the $\mathcal{X}$-frame that trajectory becomes

$$\begin{aligned} x(t) &= a + (b - v)t - \tfrac{1}{2}gt^2 \\ &\equiv a + bt - \tfrac{1}{2}gt^2 \end{aligned}$$

Manifestly, $x(t)$ and $x(t)$ are functions of the same design. But while

$$x_{\max} = a + \tfrac{1}{2}b^2/g \quad : \quad \text{occurs at time } t = b/g$$

the function $x(t)$ assumes a different maximum

$$x_{\max} = a + \tfrac{1}{2}b^2/g$$

at a different time: $t = b/g = t - v/g$. The point at issue was illustrated already in Figure 1.

Turning now from the kinematic to the Lagrangian dynamical aspects of the situation, and taking

$$L(\dot{x},x) = \tfrac{1}{2}m\dot{x}^2 - mgx \quad : \quad \text{FREE FALL LAGRANGIAN}$$

as an obvious point of departure, we construct

$$L(\dot{x},x,t) = L(\dot{x}+v, x+vt) = \tfrac{1}{2}m(\dot{x}+v)^2 - mg(x+vt)$$

which (gratifyingly, but not at all to our surprise) gives back (1)

$$\tfrac{d}{dt}m(\dot{x}+v) + \tfrac{\partial}{\partial x}mg(x+vt) = m(\ddot{x}+g) = 0$$

but does not much resemble the Lagrangian encountered at (7), from which it is seen to differ by a gauge term:

$$\big[\tfrac{1}{2}m(\dot{x}+v)^2 - mg(x+vt)\big] = \big[\tfrac{1}{2}m\dot{x}^2 - mgx\big] + \tfrac{d}{dt}\big(mvx + \tfrac{1}{2}mv^2t - \tfrac{1}{2}mgvt^2\big)$$

Evidently we must construe "Galilean transformation" to refer to a 2-STEP PROCESS

$$\left.\begin{aligned} x &\longrightarrow x = x - vt \\ L &\longrightarrow L = L + \tfrac{d}{dt}\Lambda \end{aligned}\right\} \tag{45}$$

if we are to make manifest the Galilean covariance of the Lagrangian theory of free fall. I spell out the detailed meaning and some immediate implications of the second of the preceding statements:

$$\begin{aligned} &L(\dot{x},x) \equiv \tfrac{1}{2}m\dot{x}^2 - mgx \\ &\qquad\downarrow \\ &L(\dot{x},x) = L(\dot{x}+v, x+vt) + \tfrac{d}{dt}\Lambda(x,t) \\ &\qquad\qquad\qquad\qquad \Lambda(x,t) \equiv -mvx - \tfrac{1}{2}mv^2t + \tfrac{1}{2}mgvt^2 \\ &\qquad = \tfrac{1}{2}m\dot{x}^2 - mgx \end{aligned}$$

Collaterally

$$\begin{aligned} p &\equiv \partial L/\partial \dot{x} = m\dot{x} \\ &\downarrow \\ p &\equiv \partial L/\partial \dot{x} = m\dot{x} \\ &\qquad\qquad = p - mv \end{aligned}$$

The transformation

$$\left.\begin{aligned} x &\longrightarrow x = x - vt \\ p &\longrightarrow p = p - mv \end{aligned}\right\} \tag{46}$$

is trivially canonical: $[x,p] = 1$. If we introduce the Type 2 generator

$$F_2(p,x) \equiv (x - vt)(p + mv) + f(t)$$

then we recover (42) by the Legendre process

$$\begin{aligned} x &= \partial F_2/\partial p = x - vt \\ p &= \partial F_2/\partial x = p + mv \end{aligned}$$

Moreover, we have

$$\begin{aligned} H(p,x) &= \tfrac{1}{2m}(p+mv)^2 + mg(x+vt) + \big[-v(p+mv) + f'(t)\big] \\ &= \tfrac{1}{2m}p^2 + mgx + \big[f'(t) - \tfrac{1}{2}mv^2 + mgvt\big] \\ &\downarrow \\ &= \tfrac{1}{2m}p^2 + mgx \quad \text{if we set } f(t) = \tfrac{1}{2}mv(vt - gt^2) \end{aligned}$$

Notice (compare page 15) that in the limit $v\downarrow 0$ we are again left with the familiar "Legendre generator of the identity".

If we set

$$G(p,x) = mvx - vtp \tag{47}$$

then Lie's iterative process, as described on page 16, gives

$$\begin{aligned} x(u) &= x - uvt \\ p(u) &= p - umv \end{aligned}$$

which reduces to the identity at $u = 0$ and at $u = 1$ gives back (46).

The canonical transformations (46) can be recovered (at $u - u = 1$) from the following $(u-u)$-parameterized *family* of Legendre generators

$$F(p,u;x,u) \equiv [x - vt(u-u)][p + mv(u-u)] + \tfrac{1}{2}mv^2t(u-u)^2$$

We are by now not surprised to discover (by calculation) that the generators $F(p,u;x,u)$ and $G(p,x)$ stand in the H-J relation to one another:

$$\begin{aligned} G\big(p, \tfrac{\partial F(p,u;x,u)}{\partial p}\big) - \tfrac{\partial F(p,u;x,u)}{\partial u} &= 0 \\ G\big(\tfrac{\partial F(p,u;x,u)}{\partial x}, x\big) + \tfrac{\partial F(p,u;x,u)}{\partial u} &= 0 \end{aligned}$$

Special importance will attach (quantum mechanically) to the fact that because Galilean transformations gauge the Lagrangian they gauge also the dynamical action:

$$S = S + \Lambda\Big|_{t_0}^{t_1} \quad \text{with} \quad \Lambda = -mv[x + \tfrac{1}{2}vt - \tfrac{1}{2}gt^2]$$

The point is established by computation: entrusting the details to *Mathematica*, we find

$$\begin{aligned} S(x_1 + vt_1, t_1; x_0 + vt_0, t_0) &- mv[x_1 + \tfrac{1}{2}vt_1 - \tfrac{1}{2}gt_1^2] \\ &+ mv[x_0 + \tfrac{1}{2}vt_0 - \tfrac{1}{2}gt_0^2] \\ &= \tfrac{1}{2}m\Big\{\frac{(x_1-x_0)^2}{t_1-t_0} - g(x_1+x_0)(t_1-t_0) - \tfrac{1}{12}g^2(t_1-t_0)^3\Big\} \\ &= S(x_1,t_1;x_0,t_0) \end{aligned}$$

QUANTUM DYNAMICS OF FREE FALL

13. Dimensional analysis. The adjunction of $\hbar$ to the classically available system-constants m and g permits the formation of a "natural length"

$$\ell_g \equiv \left(\tfrac{\hbar^2}{2m^2 g}\right)^{\frac{1}{3}} \equiv k^{-1} \tag{48.1}$$

whence also of a "natural energy"

$$\mathcal{E}_g \equiv mg\,\ell_g = \left(\tfrac{mg^2\hbar^2}{2}\right)^{\frac{1}{3}} \tag{48.2}$$

a "natural velocity"

$$v_g \equiv \sqrt{\tfrac{2}{m}\mathcal{E}_g} = \left(\tfrac{2g\hbar}{m}\right)^{\frac{1}{3}} \tag{48.3}$$

a "natural time"

$$\tau_g \equiv v_g/g = \left(\tfrac{2\hbar}{mg^2}\right)^{\frac{1}{3}} \tag{48.4}$$

a "natural frequency"

$$\omega_g \equiv \mathcal{E}_g/\hbar = \left(\tfrac{mg^2}{2\hbar}\right)^{\frac{1}{3}} = 1/\tau_g \tag{48.5}$$

and a "natural angular momentum" $\ell_g \cdot mv_g = \hbar$. In preceding formulæ the 2's are dimensionally inessential, but natural to the ensuing quantum mechanics.

Looking to the numerical value of ℓ_g in some typical cases, we find (when g is assigned a typical terrestrial value[14])

$$\ell_g = \begin{cases} 0.0880795 \text{ cm} & : \quad \text{electron} \\ 0.0005874 \text{ cm} & : \quad \text{proton} \end{cases}$$

The natural length is a decreasing function of mass

$$\ell_g \sim m^{-\frac{2}{3}}$$

and for macroscopic masses becomes very small indeed: for a mass of one gram we find $\ell_g \approx 10^{-21}$ cm.

The availability of a natural length permits the introduction—into the quantum theory of free fall, but not into the classical theory—of a

$$\text{dimensionless position variable} \quad : \quad z \equiv kx = x/\ell_g \tag{49}$$

and of associated dimensionless time, energy, momentum ... variables. It is in terms of those variables that we will express our quantum mechanical results. But their adoption raises a delicate point:

[14] I use *Mathematica*'s `AccelerationDueToGravity = 9.80665 m/s`^{2}.

From the physical meaning of $|\Psi(x)|^2$ we infer that dimensionally

$$[\Psi(x)] = (\text{length})^{-\frac{1}{2}}$$

By calculus

$$\int |\Psi(x)|^2\,dx = \int |\Psi(x(z))|^2\,|dx/dz|dz = \int |\Psi(\ell_g z)|^2 \ell_g dz$$

and if we would write

$$= \int |\psi(z)|^2\,dz$$

we must posit that *the wave function transforms as a scalar density of weight* $\frac{1}{2}$:

$$\psi(z) = \sqrt{\ell_g}\cdot\Psi(\ell_g z) \quad : \quad \Psi(x) = \sqrt{k}\cdot\psi(kx) \tag{50}$$

From (50) it follows that

$$[\Psi(x)] = (\text{length})^{-\frac{1}{2}} \quad\iff\quad \psi(z) \text{ is dimensionless}$$

I remark in passing that $g = GM/R^2$ where G is the gravitational constant and where M and R refer to the (gravitational) mass and mean radius of the earth. If we were concerned with the quantum physics of a "gravitationally bound Bohr atom" we would, in view of (48.1), find it natural to write

$$\ell = \left(\tfrac{\hbar^2}{2m^2 GM/\ell^2}\right)^{\frac{1}{3}}$$

which (if we discard the 2) gives

$$\ell_{\text{Bohr}} = \tfrac{\hbar^2}{GMm^2}$$

where $M \gg m$ refers now to the nuclear mass. Note the disappearance of the $\frac{1}{3}$: evidently the $\frac{1}{3}$'s in (48) are, like the ^{3} first encountered at (9), characteristic features not of gravitational physics in general but of the free fall problem in particular. But while the former arose from dimensional necessity the latter sprang from the elementary circumstance that $\int(\frac{1}{2}gt^2)\,dt = \frac{1}{6}gt^3$; *i.e.*, from the triple integration of a constant.

14. Quantum mechanical free fall according to Schrödinger. The Schrödinger equation reads

$$\Big\{-\tfrac{\hbar^2}{2m}\big(\tfrac{\partial}{\partial x}\big)^2 + mgx\Big\}\Psi(x,t) = i\hbar\tfrac{\partial}{\partial t}\Psi(x,t) \tag{51}$$

The basic problem is to display the normalized solution of (51)

$$\int_{-\infty}^{+\infty} |\Psi(x,t)|^2\,dx = 1$$

that conforms to the prescribed initial data $\Psi(x,t_0)$.

If $\Psi(x,t)$ is assumed to possess the separated structure

$$\Psi(x,t) = \Psi(x)\cdot e^{-\frac{i}{\hbar}E\,t}$$

then (51) requires that $\Psi(x)$ be a solution of the t-independent Schrödinger equation

$$\Big\{-\tfrac{\hbar^2}{2m}\big(\tfrac{d}{dx}\big)^2 + mg\,x\Big\}\Psi(x) = E\Psi(x) \tag{52}$$

which will serve as our point of departure.

Let (52) be written

$$\big(\tfrac{d}{dx}\big)^2\Psi(x) = \tfrac{2m^2g}{\hbar^2}\big(x - \tfrac{E}{mg}\big)\Psi(x)$$

Now introduce the shifted/rescaled new independent variable $y = k\big(x - \frac{E}{mg}\big)$ and adopt the notation $\Psi(x) = \sqrt{k}\cdot\psi(y)$. The differential equation then becomes

$$\big(\tfrac{d}{dy}\big)^2\psi(y) = \tfrac{2m^2g}{\hbar^2}k^{-3}y\,\psi(y)$$

which motivates us—as previously we were dimensionally motivated—to assign to k the particular value $k = \big(\frac{2m^2g}{\hbar^2}\big)^{\frac{1}{3}}$. The time-independent Schrödinger equation is brought thus to the strikingly simple form

$$\big(\tfrac{d}{dy}\big)^2\psi(y) = y\,\psi(y) \tag{53}$$

where—remarkably—<u>the value of E has been absorbed into the definition of the independent variable</u>:

$$\begin{aligned} y &\equiv \big(\tfrac{2m^2g}{\hbar^2}\big)^{\frac{1}{3}}\big(x - \tfrac{E}{mg}\big)\\ &= k(x-a)\\ &\qquad\quad a \equiv \tfrac{E}{mg} = \begin{cases}\text{maximal height achieved by a}\\ \text{particle lofted with energy } E\end{cases}\\ &\equiv z - \alpha \end{aligned} \tag{54}$$

Here a describes the classical "turning point" of the trajectory pursued by a particle with energy E (see again Figure 1), and $\alpha \equiv ka$ provides a dimensionless description of the turning point. We are, by the way, in position now to understand why the anticipatory 2's were introduced into the definitions (48).

At (53) we have *Airy's differential equation*, first encountered in George Airy's "Intensity of light in the neighborhood of a caustic" (1838).[15] The

[15] At this point I begin to borrow directly from material that begins on page 22 of Chapter 2: "Weyl Transform and the Phase Space Formalism" in ADVANCED QUANTUM TOPICS (2000).

solutions are linear combinations of the *Airy functions* $\text{Ai}(y)$ and $\text{Bi}(y)$, which are close relatives of the Bessel functions of orders $\pm\frac{1}{3}$, and of which (since $\text{Bi}(y)$ diverges as $y \to \infty$) only the former

$$\text{Ai}(y) \equiv \tfrac{1}{\pi}\int_0^\infty \cos\left(yu + \tfrac{1}{3}u^3\right) du \tag{55}$$

will concern us.[16] To gain insight into the origin of Airy's construction, write

$$f(y) = \tfrac{1}{2\pi}\int_{-\infty}^{+\infty} g(u)e^{iyu}\, du$$

and notice that $f'' - yf = 0$ entails

$$\tfrac{1}{2\pi}\int_{-\infty}^{+\infty} \left[-\,u^2 g(u) + ig(u)\tfrac{d}{du}\right]e^{iyu}\, du = 0$$

Integration by parts gives

$$\tfrac{1}{2\pi}ig(u)e^{iyu}\Big|_{-\infty}^{+\infty} - \tfrac{1}{2\pi}\int_{-\infty}^{+\infty} \left[u^2 g(u) + ig'(u)\right]e^{iyu}\, du = 0$$

The leading term vanishes if we require $g(\pm\infty) = 0$. We are left then with a first-order differential equation $u^2 g(u) + ig'(u) = 0$ of which the general solution is $g(u) = A\cdot e^{i\frac{1}{3}u^3}$. So we have

$$f(y) = A\cdot\tfrac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\,(yu+\frac{1}{3}u^3)}\, du = A\cdot\tfrac{1}{\pi}\int_0^\infty \cos\left(yu + \tfrac{1}{3}u^3\right) du$$

It was to achieve

$$\int_{-\infty}^{+\infty} \text{Ai}(y)\, dy = 1 \tag{56}$$

that Airy assigned the value $A = 1$ to the constant of integration.

---

[16] For summaries of the properties of Airy functions see Chapter 56 in Spanier & Oldham, *An Atlas of Functions* (1987) or §10.4 in Abramowitz & Stagun, *Handbook of Mathematical Functions* (1965). Those functions are made familiar to students of quantum mechanics by their occurance in the "connection formulæ" of simple WKB approximation theory: see, for example, Griffiths' §8.3, or C. M. Bender & S. A. Orszag, *Advanced Mathematical Methods for Scientists & Engineers* (1978), §10.4.

Returning with this mathematics to the quantum physics of free fall, we see that solutions of the Schrödinger equation (52) can—in physical variables—be described

$$\Psi_E(x) = (\text{normalization factor}) \cdot \text{Ai}\big(k(x - a_E)\big) \tag{57.1}$$

where again

$$a_E \equiv \tfrac{E}{mg} = \text{classical turning point of a particle lofted with energy } E$$

In dimensionless variables (57.1) becomes

$$\psi_{\mathcal{E}}(z) = N \cdot \text{Ai}(z - \alpha_{\mathcal{E}}) \tag{57.2}$$

where N is a normalization factor, soon to be determined.

It is a striking fact—evident in (57)—that the eigenfunctions $\psi_{\mathcal{E}}(z)$ *all have the same shape* (*i.e.*, are translates of one another: see Figure 5), and remarkable also that the *the energy spectrum is continuous, and has no least member*: the system possesses no ground state. One might dismiss this highly unusual circumstance as an artifact, attributable to the physical absurdity of the idealized free-fall potential

$$U(x) = mgx \quad : \quad -\infty < x < +\infty$$

but then has to view with surprise the major qualitative difference between the cases $g \neq 0$ (no ground state) and $g = 0$ (ground state abruptly springs into existence). I prefer to adopt the notion that *"free fall" is free motion relative to a non-inertial frame*, and to trace that "major qualitative difference" to the major difference between being/not being inertial.

The eigenfunctions $\Psi_E(x)$ share with the free particle functions $e^{\pm\frac{i}{\hbar}\sqrt{2mE}\,x}$ the property that they are not individually normalizable,[17] but require assembly into "wavepackets." They do, however, comprise a complete orthonormal set, in the sense which I digress now to establish. Let

$$f(z, \alpha) \equiv \text{Ai}(z - \alpha)$$

To ask of the α-indexed functions $f(z, \alpha)$

- Are they *orthonormal*: $\int f(z,\alpha) f(z,\beta)\, dy = \delta(\alpha - \beta)$?
- Are they *complete*: $\int f(y,\alpha) f(z,\alpha)\, d\alpha = \delta(y - z)$?

is, in fact, to ask the same question twice, for both are notational variants of this question: Does

$$\int_{-\infty}^{+\infty} \text{Ai}(z-\alpha)\,\text{Ai}(z-\beta)\, dz = \delta(\alpha - \beta)\,?$$

[17] Asymptotically $\text{Ai}^2(y) \sim \frac{1}{\pi\sqrt{|y|}} \sin^2\left(\frac{2}{3}|y|^{\frac{3}{2}} + \frac{\pi}{4}\right)$ dies as $y \downarrow -\infty$, but so slowly that the limit of $\int_y^0 \text{Ai}^2(u)\, du$ blows up.

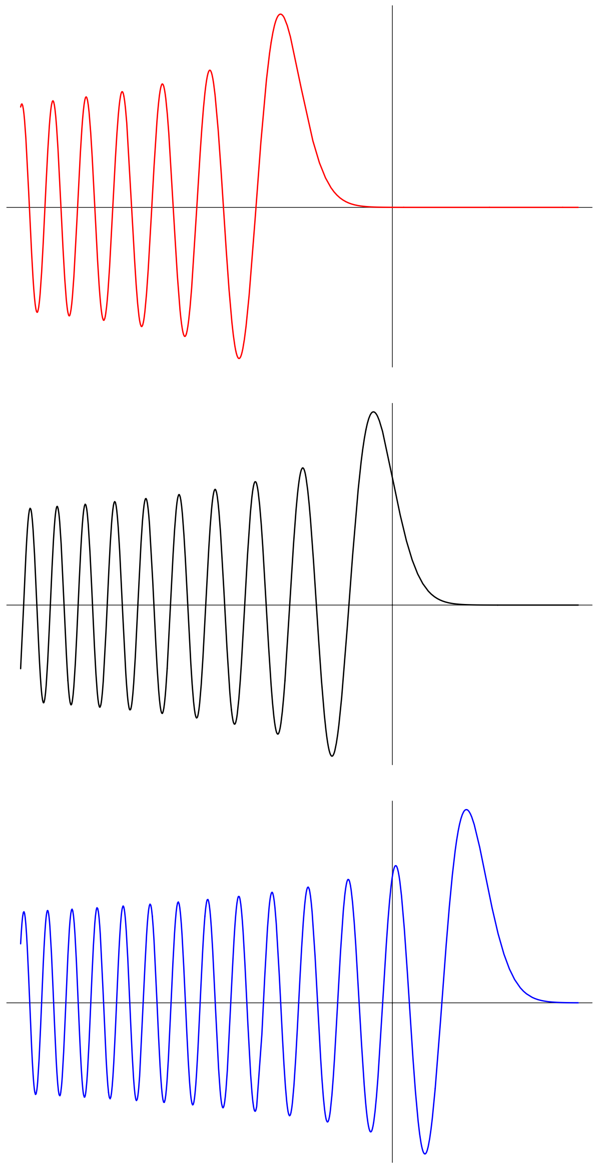

FIGURE 5: *Free fall eigenfunctions* $\psi_{\mathcal{E}}(z)$ *with* $\mathcal{E} < 0$, $\mathcal{E} = 0$, $\mathcal{E} > 0$, *in descending order. The remarkable translational similarity of the eigenfunctions can be understood as a quantum manifestation of the self-similarity evident at several points already in the classical physics of free fall (see again §3 and Figure 4).*

An affirmative answer (which brings into being a lovely "Airy-flavored Fourier analysis") is obtained as follows:

$$
\begin{aligned}
\int_{-\infty}^{+\infty} &\mathrm{Ai}(z-\alpha)\,\mathrm{Ai}(z-\beta)\,dz \\
&= \big(\tfrac{1}{2\pi}\big)^2 \iiint e^{i\,[(z-\alpha)u+\frac{1}{3}u^3]} e^{i\,[(z-\beta)v+\frac{1}{3}v^3]}\,dudvdz \\
&= \tfrac{1}{2\pi}\iint e^{i\frac{1}{3}(u^3+v^3)} e^{-i(\alpha u+\beta v)} \underbrace{\Big\{\tfrac{1}{2\pi}\int e^{iz(u+v)}\,dz\Big\}}_{\delta(u+v)} dudv \\
&= \tfrac{1}{2\pi}\int \underbrace{e^{i\frac{1}{3}(v^3-v^3)}}_{1} e^{iv(\alpha-\beta)}\,dv = \delta(\alpha-\beta)
\end{aligned}
$$

So for our free fall wave functions we have "orthogonality in the sense of Dirac:"

$$
\begin{aligned}
\int_{-\infty}^{+\infty} \psi^*_{\mathcal{E}'}(z)\,\psi_{\mathcal{E}''}(z)\,dz &= N^2\int_{-\infty}^{+\infty}\mathrm{Ai}\big(z-\alpha_{\mathcal{E}'}\big)\,\mathrm{Ai}\big(z-\alpha_{\mathcal{E}''}\big)\,dz \\
&= N^2\cdot\delta(\mathcal{E}'-\mathcal{E}'') \\
&\downarrow \\
&= \delta(\mathcal{E}'-\mathcal{E}'') \quad \text{provided we set } N=1
\end{aligned}
\tag{58.1}
$$

The functions thus normalized are complete in the sense that

$$
\int_{-\infty}^{+\infty} \psi^*_{\mathcal{E}}(z')\,\psi_{\mathcal{E}}(z'')\,d\mathcal{E} = \delta(z'-z'') \tag{58.2}
$$

15. Construction & structure of the free fall propagator. Quite generally (subject only to the assumption that the Hamiltonian is t-independent), eigenvalues E_n and eigenfunctions $\Psi_n(x)$, when assembled to produce the "propagator"

$$
K(x_1,t_1;x_0,t_0) \equiv \sum_n \Psi_n(x_1)\Psi^*_n(x_0)e^{-\frac{i}{\hbar}E_n(t_1-t_0)} \tag{59}
$$

permit one to describe the dynamical evolution of any prescribed initial state:

$$
\Psi(x,t_0) \longmapsto \Psi(x,t) = \int K(x,t;x_0,t_0)\Psi(x_0,t_0)\,dx_0 \tag{60}
$$

The propagator, looked upon as an (x_0,t_0)-parameterized function of x and t, is a solution of the t-dependent Schrödinger equation, distinguished from other solutions by the property

$$
\lim_{t\downarrow t_0} K(x,t;x_0,t_0) = \delta(x-x_0)
$$

In the present context (59) supplies

$$\mathcal{K}(z,t;z_0,0)=\int_{-\infty}^{+\infty}\psi_{\mathcal{E}}(z)\psi^*_{\mathcal{E}}(z_0)e^{-i\mathcal{E}\theta}\,d\mathcal{E}$$

where I have elected to work in dimensionless variables, noting that

$$\begin{aligned} E=\mathcal{E}_g\cdot\mathcal{E}= &\quad(mg\,\ell_g)\cdot\mathcal{E} \;:\; \begin{cases}\text{relates physical energy } E\\ \text{to dimensionless energy } \mathcal{E}\end{cases}\\ t=\;\tau_g\cdot\theta=&(\hbar/mg\,\ell_g)\cdot\theta \;:\; \begin{cases}\text{relates physical time } t\\ \text{to dimensionless time } \theta\end{cases}\end{aligned}$$

entail

$$\tfrac{1}{\hbar}Et=\mathcal{E}\theta$$

Notice also that the relationship between the energy E and the turning point a, when expressed in terms of dimensionless variables, becomes

$$mg(\ell_g\alpha)=(mg\ell_g)\mathcal{E}\quad:\quad\text{reduces to }\alpha=\mathcal{E}$$

of which we will make free use. Working from

$$\psi_{\mathcal{E}}(x)=\text{Ai}\big(z-\alpha\big)=\tfrac{1}{2\pi}\int_{-\infty}^{+\infty}e^{i\,(\frac{1}{3}u^3+[z-\alpha]u)}\,du$$

we have[18]

$$\begin{aligned}\mathcal{K}&=\big(\tfrac{1}{2\pi}\big)^2\iiint e^{i\,(\frac{1}{3}u^3+[z-\alpha]u)}e^{i\,(\frac{1}{3}v^3+[z_0-\alpha]v)}e^{-i\alpha\theta}\,dudvd\alpha\\ &=\big(\tfrac{1}{2\pi}\big)^2\iiint e^{i\,\frac{1}{3}(u^3+v^3)}e^{i(zu+z_0v)}e^{-i(u+v+\theta)\alpha}\,dadudv\\ &=\tfrac{1}{2\pi}\iint e^{i\,\frac{1}{3}(u^3+v^3)}e^{i(zu+z_0v)}\delta(u+v+\theta)\,dudv\\ &=\tfrac{1}{2\pi}\int e^{i\,\frac{1}{3}[v^3-(v+\theta)^3]}e^{i[z_0v-z(v+\theta)]}\,dv\end{aligned}$$

But $v^3-(v+\theta)^3=-3v^2\theta-3v\theta^2-\theta^3$ so

$$=\tfrac{1}{2\pi}e^{-i(\frac{1}{3}\theta^3+z\theta)}\cdot\int e^{-i\theta v^2-i[\theta^2+(z-z_0)]}\,dv$$

The sole surviving integral is (formally) Gaussian, and its elementary evaluation supplies

$$\begin{aligned}\mathcal{K}&=\tfrac{1}{2\pi}e^{-i(\frac{1}{3}\theta^3+z\theta)}\cdot\sqrt{\tfrac{2\pi}{2i\theta}}e^{-\frac{1}{4i\theta}[\theta^2+(z-z_0)]^2}\\ &=\sqrt{\tfrac{1}{4\pi i\theta}}\exp\Big\{i\Big[\frac{(z-z_0)^2}{4\theta}+\big[\tfrac{1}{2}(z-z_0)-z\big]\theta+\big[\tfrac{1}{4}-\tfrac{1}{3}\big]\theta^3\Big]\Big\}\end{aligned}$$

---

[18] The following argument was adapted from page 32 of some informal notes "Classical/quantum mechanics of a bouncing ball" (March 1994) that record the results of my first excursion into this problem area.

Translation to physical variables gives

$$\begin{aligned} K &= k \cdot \mathcal{K} \\ &= \sqrt{\tfrac{m}{2\pi i\hbar t}}\,\exp\Big\{\tfrac{i}{\hbar}\Big[\tfrac{m}{2t}(x-x_0)^2 - \tfrac{1}{2}mg(x+x_0)\,t - \tfrac{1}{24}mg^2t^3\Big]\Big\} \end{aligned} \tag{61}$$

Remarkably, this result (after a trivial notational adjustment: $t \mapsto t_1 - t_0$) can be expressed

$$K(x_1,t_1;x_0,t_0) = \sqrt{\tfrac{m}{2\pi i\hbar(t_1-t_0)}} \cdot e^{\frac{i}{\hbar}S(x_1,t_1;x_0,t_0)} \tag{62}$$

where $S(x_1,t_1;x_0,t_0)$ is precisely the classical free-fall action function, first encountered at (9). Some theoretical importance attaches also to the fact that

$$\sqrt{\tfrac{m}{2\pi i\hbar(t_1-t_0)}} \quad \text{can be written} \quad \sqrt{\tfrac{i}{\hbar}\tfrac{\partial^2 S}{\partial x_1 \partial x_0}}$$

By way of commentary: if, into the Schrödinger equation (51), we insert

$$\Psi = A\,e^{\frac{i}{\hbar}S}$$

we obtain

$$A\Big\{\tfrac{1}{2m}(S_x)^2 + mgx + S_t\Big\} - i\hbar\underbrace{\Big\{\tfrac{1}{2m}\big[AS_{xx} + 2A_xS_x\big] + A_t\Big\}}_{=\frac{1}{2A}\big\{(\frac{1}{m}S_xA^2)_x + (A^2)_t\big\}} - \hbar^2\tfrac{1}{2m}A_{xx} = 0$$

Returning with this information to (62) we find that

- the leading $\{\}$ vanishes because S satisfies the Hamilton-Jacobi equation
- the final term vanishes because A is x-independent
- the the middle $\{\}$—which brings to mind a "continuity equation," of precisely the sort that $A^2 = |\psi|^2$ is known in quantum mechanics to satisfy (as a expression of the "conservation of probability")—vanishes by computation:

$$\Big(\tfrac{1}{m}\big[\tfrac{m}{t}(x-x_0) - \tfrac{1}{2}mgt\big]\tfrac{1}{t}\Big)_x + \Big(\tfrac{1}{t}\Big)_t = \tfrac{1}{t^2} - \tfrac{1}{t^2} = 0$$

16. Dropped Gaussian wavepacket. Let ψ be given initially by

$$\psi(z,0) = \tfrac{1}{\sqrt{\sigma\sqrt{2\pi}}}\,e^{-\frac{1}{4}[z/\sigma]^2} \tag{63}$$

Then

$$|\psi(z,0)|^2 = \tfrac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}[z/\sigma]^2}$$

is a normalized Gaussian

$$\int_{-\infty}^{+\infty} |\psi(z,0)|^2\,dz = 1$$

with second moment

$$\langle z^2\rangle = \int_{-\infty}^{+\infty} z^2|\psi(z,0)|^2\,dz = \sigma^2$$

Returning with (61) and (63) to (60), we confront the messy Gaussian integral

$$\psi(z,\theta) = \sqrt{\tfrac{1}{4\pi i\theta}\tfrac{1}{\sigma\sqrt{2\pi}}}\int \exp\Big\{i\Big[\tfrac{1}{4\theta}(z-z_0)^2 - \tfrac{1}{2}(z+z_0)\theta - \tfrac{1}{12}\theta^3\Big] - \tfrac{1}{4\sigma^2}z_0^2\Big\}dz_0$$

Mathematica responds with a result

$$= \sqrt{\tfrac{1}{4\pi i\theta}\tfrac{1}{\sigma\sqrt{2\pi}}}\,2\sqrt{\pi}\Big[\tfrac{1}{\sigma^2} - i\tfrac{1}{\theta}\Big]^{-\frac{1}{2}}\cdot\exp\Big\{\tfrac{1}{12}\tfrac{6\theta^2 z-3z^2+\theta^4-12i\sigma^2\theta z-4i\sigma^2\theta^3}{\sigma^2+i\theta}\Big\}$$

which we have now to disentangle. Look first to the prefactor, which after simplifications becomes

$$\begin{aligned}\sqrt{\tfrac{1}{4\pi i\theta}\tfrac{1}{\sigma\sqrt{2\pi}}}\,2\sqrt{\pi}\Big[\tfrac{1}{\sigma^2} - i\tfrac{1}{\theta}\Big]^{-\frac{1}{2}} &= \sqrt{\tfrac{1}{\Sigma(t)\sqrt{2\pi}}}\\ \Sigma(t) &\equiv \sigma\big[1+i\tfrac{\theta}{\sigma^2}\big]\\ &= \sigma\sqrt{1+\big(\tfrac{\theta}{\sigma^2}\big)^2}\cdot e^{i\,\delta(\theta)}\\ &\equiv \sigma(\theta)\cdot e^{i\,\delta(\theta)}\end{aligned}$$

with $\delta(\theta)\equiv\arctan(\theta/\sigma^2)$. Look next to the argument of the exponential: we find

$$\Big\{\text{etc.}\Big\} = -\tfrac{[z+\theta^2]^2}{4\sigma^2(\theta)} - i\Delta(z,\theta)$$

where $\Delta(z,\theta)$ is a complicated term the details of which need not concern us. We now have

$$\psi(z,\theta) = \frac{1}{\sqrt{\sigma(\theta)\sqrt{2\pi}}}\,\exp\Big\{-\tfrac{1}{4}\Big[\frac{z+\theta^2}{\sigma(\theta)}\Big]^2 - i\,\phi(x,t)\Big\}$$

with $\phi\equiv\frac{1}{2}\delta+\beta$. The phase factor is of no present interest because it disappears when we look to the probability density

$$|\psi(z,\theta)|^2 = \frac{1}{\sigma(\theta)\sqrt{2\pi}}\,\exp\Big\{-\tfrac{1}{2}\Big[\frac{z+\theta^2}{\sigma(\theta)}\Big]^2\Big\} \tag{64.1}$$

If, in place of (63), we had written

$$\psi(z,0) = e^{i\rho z}\cdot\frac{1}{\sqrt{\sigma\sqrt{2\pi}}}\,e^{-\frac{1}{4}[z/\sigma]^2}$$

then in place of (64.1) we would have obtained

$$|\psi(z,\theta)|^2 = \frac{1}{\sigma(\theta)\sqrt{2\pi}}\,\exp\Big\{-\tfrac{1}{2}\Big[\frac{z-2\rho\theta+\theta^2}{\sigma(\theta)}\Big]^2\Big\} \tag{64.2}$$

In physical variables[19] equations (64) become

$$|\Psi(x,t)|^2 = \begin{cases} \frac{1}{s(t)\sqrt{2\pi}}\exp\Big\{-\frac{1}{2}\Big[\frac{x+\frac{1}{2}gt^2}{s(t)}\Big]^2\Big\} & : \text{ DROPPED GAUSSIAN} \\ \\ \frac{1}{s(t)\sqrt{2\pi}}\exp\Big\{-\frac{1}{2}\Big[\frac{x-vt+\frac{1}{2}gt^2}{s(t)}\Big]^2\Big\} & : \text{ LOFTED GAUSSIAN} \end{cases} \tag{65.1}$$

Equations (65) describe Gaussian distributions the centers of which move ballistically, but which disperse hyperbolically[20]—*just what you would expect to see if the free particle result standard to the textbooks were viewed from a uniformly accelerated frame*.

17. Other dropped wavefunctions. What happens when you drop the familiar free particle eigenfunction

$$\Psi(x,0) = \tfrac{1}{\sqrt{h}}e^{\frac{i}{\hbar}px} \quad : \quad p \text{ an adjustable real constant}$$

Immediately

$$\begin{aligned}\Psi(x,t) &= \sqrt{\tfrac{m}{iht}\tfrac{1}{h}}\int \exp\Big\{\tfrac{i}{\hbar}\Big[\tfrac{m}{2t}(x-x_0)^2 - \tfrac{1}{2}mg(x+x_0)t \\ &\qquad\qquad - \tfrac{1}{24}mg^2t^3\Big] + \tfrac{i}{\hbar}px_0\Big\}\,dx_0 \\ &= \tfrac{1}{\sqrt{h}}\exp\Big\{\tfrac{i}{\hbar}\Big[px - \tfrac{1}{2m}p^2t - g\big\{mxt - \tfrac{1}{2}pt^2 + \tfrac{1}{6}mgt^3\big\}\Big]\Big\} \end{aligned} \tag{66}$$

I have not been able to account term-by-term in an intuitively satisfying way for the design of this result (see, however, §25 below), but have verified that the function $S(x,t;p) \equiv \big[\text{etc.}\big]$ does in fact satisfy the Hamilton-Jacobi equation (equivalently: $\Psi(x,t)$ does satisfy the Schrödinger equation).

---

[19] Recall from (50) that $|\Psi|^2 = k|\psi|^2$, define

$$s \equiv \ell_g\sigma \quad \text{and} \quad s(t) \equiv \ell_g\sigma(\theta) = s\sqrt{1+\big(\tfrac{\hbar t}{2ms^2}\big)^2} \tag{65.2}$$

and notice that $\theta^2/\sigma = (mgt/\hbar k)^2/ks = \frac{1}{2}gt^2$. Also write $\rho z = \frac{1}{\hbar}mvx$ to obtain $\rho = mv\ell_g/\hbar$ (dimensionless momentum), giving $2\rho\theta/\sigma = vt/s$.

[20] I say "hyperbolically" because

$$\sigma(t) = \sigma_0\sqrt{1+(t/\tau)^2} \quad : \quad \tau \equiv 2m\sigma_0^2/\hbar$$

can be written

$$(\sigma/\sigma_0)^2 - (t/\tau)^2 = 1$$

Note that the dispersion law is g-independent.

It is a curious (though self-evident!) fact that when you drop a free-fall eigenfunction

$$\psi(x,0) \equiv \psi_{\mathcal{E}}(x)$$

it does not fall: it simply "stands there (levitates!) and buzzes"

$$\psi(x,0) \quad\longrightarrow\quad \psi(x,t) = \psi(x,0)\cdot e^{-\frac{i}{\hbar}E(\mathcal{E})t} \tag{67}$$

I have described elsewhere a "quantum calculus of moments"[21] that can be used to show—very simply—that the phenomenon illustrated at (65) is in fact not special to Gaussian wavepackets: *every* wavepacket falls in such a way that $\langle x\rangle$ moves ballistically, and Δx grows hyperbolically. Equations (66) and (67) do not provide counterexamples, for they refer to wavefunctions that are not normalizable, do not describe quantum states—are, in short, not wavepackets.

18. Ladder operators for the free fall problem. The time-independent Schrödinger equation (48) is a particular representation of the abstract equation

$$\mathsf{H}|E) = E|E) \quad\text{with}\quad \mathsf{H} \equiv \tfrac{1}{2m}\mathsf{p}^2 + mg\mathsf{x} \tag{68}$$

From the fundamental commutation relation $[\mathsf{x},\mathsf{p}] = i\hbar\mathsf{I}$ it follows familiarly[22] that if $\mathsf{T}(\xi)$ refers to a member of the ξ-parameterized population of unitary operators

$$\mathsf{T}(\xi) \equiv e^{-\frac{i}{\hbar}\xi\,\mathsf{p}}$$

then

$$\left.\begin{aligned}\mathsf{x}\,\mathsf{T}(\xi) &= \mathsf{T}(\xi)(\mathsf{x}+\xi\,\mathsf{I})\\ \mathsf{p}\,\mathsf{T}(\xi) &= \mathsf{T}(\xi)\;\;\mathsf{p}\end{aligned}\right\} \tag{69}$$

It follows that if $\mathsf{x}|x) = x|x)$ then $\mathsf{x}\,\mathsf{T}(\xi)|x) = (x+\xi)\,\mathsf{T}(\xi)|x)$; in other words

$$\mathsf{T}(\xi)|x) = |x+\xi) \quad : \quad \mathsf{T}(\xi) \textit{ translates} \text{ with respect to the } \mathsf{x}\text{-spectrum}$$

But because x enters linearly into the design of the free fall Hamiltonian we have

$$\mathsf{H}\,\mathsf{T}(\xi) = \mathsf{T}(\xi)(\mathsf{H} + mg\xi\,\mathsf{I})$$

which when applied to $|E)$ gives

$$\mathsf{H}\,\mathsf{T}(\xi)|E) = (E + mg\xi)\,\mathsf{T}(\xi)|E)$$

In short:

$$\mathsf{T}(\xi)|E) = |E + mg\xi) \;:\; \mathsf{T}(\xi) \text{ also } \textit{translates} \text{ with respect to the } \mathsf{H}\text{-spectrum}$$

---

[21] ADVANCED QUANTUM TOPICS (2000) Chapter 2, page 51. The method is an elaboration of the idea central to Ehrenfest's theorem.

[22] See Chapter 0, page 34 in the notes just cited.

Using these two facts together, we find that if

$$\Psi_E(x) \equiv (x|E)$$

then

$$\Psi_{E+mga}(x) = (x|\,\mathbf{T}(a)|E) = (x-a|E) = \Psi_E(x-a) \tag{70}$$

We have thus accounted abstractly for a remarkable property of the free fall eigenfunctions that was previously[23] obtained analytically from the theory of Airy functions. The defect of the argument is that it provides no hint that $\Psi_E(x)$ has anything to do with Airy functions, no indication of what $|\Psi_E(x)|^2$ might look like when plotted.

The ladder operator technique is most familiar as encountered in the quantum theory of the harmonic oscillator and the quantum theory of angular momentum. We have seen that it enters also quite naturally into the quantum theory of unobstructed free fall . . . but that in the latter application *the ladder has continuously indexed rungs*! And no bottom rung!!

19. Schrödinger equation in the momentum representation. Define

$$\Phi_E(p) \equiv (p|E)$$

and observe that in the momentum representation the time-independent Schrödinger equation (68) reads

$$\left\{\tfrac{1}{2m}p^2 - mg\big(\tfrac{\hbar}{i}\tfrac{d}{dp}\big)\right\}\Phi_E = E\,\Phi_E \tag{71}$$

or again (and more conveniently)

$$\tfrac{d}{dp}\Phi_E = i\big(\tfrac{1}{2m^2g\hbar}p^2 - \tfrac{1}{mg\hbar}E\big)\Phi_E$$

This is an ordinary differential equation *of first order*, and its solutions are immediate:

$$\Phi_E(p) = A\cdot\exp\Big\{i\big(\tfrac{1}{6m^2g\hbar}p^3 - \tfrac{1}{mg\hbar}Ep\big)\Big\}$$

where A is an arbitrary complex constant. From this it follows[24] that

$$\begin{aligned}\Psi_E(x) &= \tfrac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} e^{\frac{i}{\hbar}p\,x}\Phi_E(p)\,dp\\ &= \tfrac{1}{\sqrt{2\pi\hbar}}2A\int_0^\infty \cos\Big\{\big(\tfrac{mgx-E}{mg\hbar}\big)p + \tfrac{1}{3}\big(\tfrac{1}{2m^2g\hbar}p^3\big)\Big\}\,dp\\ &= \tfrac{1}{\sqrt{2\pi\hbar}}(2m^2g\hbar)^{\frac{1}{3}}2\pi A\cdot\tfrac{1}{\pi}\int_0^\infty \cos\Big\{yu + \tfrac{1}{3}u^3\Big\}\,du\end{aligned}$$

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[23] See again page 26 and Figure 5.
[24] See (82) on page 36 of Chapter 0 in ADVANCED QUANTUM TOPICS (2000).

where $u \equiv p/(2m^2g\hbar)^{\frac{1}{3}}$ and where $y \equiv \big(\frac{mgx-E}{mg\hbar}\big)(2m^2g\hbar)^{\frac{1}{3}}$, in the notations introduced at (48), can be described $y = k(x - E/mg) = k(x - a_E)$. Drawing now upon the definition (55) of the Airy function, we have

$$\Psi_E(x) = \tfrac{1}{\sqrt{2\pi\hbar}}(2m^2g\hbar)^{\frac{1}{3}}2\pi A\cdot \mathrm{Ai}\big(k(x-a_E)\big)$$

and to achieve detailed agreement with (50)/(58.1)—*i.e.*, to achieve (in the sense of Dirac) ortho*normality* of the eigenfunctions—we assign to A the value that gives $\frac{1}{\sqrt{2\pi\hbar}}(2m^2g\hbar)^{\frac{1}{3}}2\pi A = \sqrt{k} = (2m^2g/\hbar^2)^{\frac{1}{6}}$. Which is to say: we set

$$A = (2\pi\wp_g)^{-\frac{1}{2}} \quad \text{where} \quad \wp_g \equiv k\hbar = \hbar/\ell_g = (2m^2g\hbar)^{\frac{1}{3}}$$

defines the "natural momentum" that might appropriately be added to the list (48).

Thus *by paraphrase of the argument encountered already on page 25* does the Airy function make its entrance when (as apparently Landau was the first to do) one elects to use (not the x-representation but) the p-representation in approaching the quantum mechanics of free fall.

The time-dependent companion of (71) can be written

$$\big\{i\hbar\tfrac{\partial}{\partial t} - \tfrac{1}{2m}p^2 - img\hbar\tfrac{\partial}{\partial p}\big\}\mathit{\Phi}(p,t) = 0$$

or again

$$\big\{\tfrac{\partial}{\partial p} - i\tfrac{1}{2m^2g\hbar}p^2 - \tfrac{1}{mg}\tfrac{\partial}{\partial t}\big\}\mathit{\Phi}(p,t) = 0 \tag{72}$$

We have, however, the operator identity ("shift rule")

$$\big\{\tfrac{\partial}{\partial p} - i\tfrac{1}{2m^2g\hbar}p^2 - \tfrac{1}{mg}\tfrac{\partial}{\partial t}\big\} = e^{+i\frac{1}{3}(p^3/2m^2g\hbar)}\big\{\tfrac{\partial}{\partial p} - \tfrac{1}{mg}\tfrac{\partial}{\partial t}\big\}e^{-i\frac{1}{3}(p^3/2m^2g\hbar)}$$

so (72) can be rendered

$$\big\{\tfrac{\partial}{\partial p} - \tfrac{1}{mg}\tfrac{\partial}{\partial t}\big\}F(p,t) = 0 \quad \text{with} \quad F(p,t) \equiv e^{-i\frac{1}{3}(p^3/2m^2g\hbar)}\cdot\mathit{\Phi}(p,t)$$

This (compare (51)) is a partial differential equation *of first order*, and its general solution is immediate:

$$F(p,t) = f(p+mg\,t) \quad : \quad f(\bullet)\text{ arbitrary}$$

The general solution of (72) can therefore be described

$$\mathit{\Phi}(p,t) = f(p+mg\,t)\cdot e^{+i\frac{1}{3}(p/\wp)^3} \tag{73}$$

and gives

$$\Psi(x,t) = \tfrac{1}{\sqrt{2\pi\hbar}}\int f(p+mg\,t)\,e^{i[\frac{1}{\hbar}p\,x+\frac{1}{3}(p/\wp)^3]}\,dp \tag{74.1}$$

$$= \tfrac{1}{\sqrt{2\pi\hbar}}\int f(q)\exp\Big\{i\Big[\tfrac{1}{\hbar}(q-mg\,t)x + \tfrac{1}{3\wp^3}(q-mg\,t)^3\Big]\Big\}\,dq \tag{74.2}$$

From $\frac{1}{2\pi\hbar}\int e^{\frac{i}{\hbar}(p-q)x}\,dx = \delta(p-q)$ it follows almost immediately that

$$\int \Psi^*(x)\Psi(x)\,dx = 1 \quad\Longleftrightarrow\quad \int f^*(q)f(q)\,dq = 1 \tag{75}$$

Equations (73) & (74) are, as it happens, central to a recent paper by Miki Wadati,[25] and it is to Wadati that we owe the following observations:

- Set $f(q) = F\delta(q)$ and obtain what Wadati calls the "plane wave solution"

$$\begin{aligned}\Psi(x,t) &= \tfrac{1}{\sqrt{2\pi\hbar}}F\exp\Big\{i\Big[-\tfrac{1}{\hbar}mgxt - \tfrac{1}{3\wp^3}(mgt)^3\Big]\Big\}\\ &= \tfrac{1}{\sqrt{2\pi\hbar}}F\exp\Big\{-\tfrac{i}{\hbar}mg\big[x+\tfrac{1}{6}gt^2\big]t\Big\}\end{aligned}$$

The placement of the t's is a bit strange, but calculation confirms that in fact $-\frac{\hbar^2}{2m}\Psi_{xx} + mgx\Psi - i\hbar\Psi_t = 0$. Necessarily $[F] = (\text{momentum})^{\frac{1}{2}}$.

- Set $f(q) = F\exp\big\{-\frac{i}{\hbar}\frac{E}{mg}q\big\}$ and obtain

$$\begin{aligned}\Psi(x,t) &= \tfrac{1}{\sqrt{2\pi\hbar}}\int F\exp\Big\{-\tfrac{i}{\hbar}\tfrac{E}{mg}(p+mgt)\Big\}\exp\Big\{i\Big[\tfrac{1}{\hbar}px + \tfrac{1}{3\wp^3}p^3\Big]\Big\}\,dp\\ &= e^{-\frac{i}{\hbar}Et}\cdot F\tfrac{1}{\sqrt{2\pi\hbar}}\wp\int_{-\infty}^{+\infty}\exp\Big\{i\big[k(x-\tfrac{E}{mg})u + \tfrac{1}{3}u^3\big]\Big\}\,du \quad : \quad u \equiv p/\wp\\ &= e^{-\frac{i}{\hbar}Et}\cdot F\tfrac{1}{\sqrt{2\pi\hbar}}\wp 2\pi\cdot\tfrac{1}{\pi}\int_0^\infty \cos\Big\{i\big[k(x-\tfrac{E}{mg})u + \tfrac{1}{3}u^3\big]\Big\}\,du\\ &= e^{-\frac{i}{\hbar}Et}\cdot F\tfrac{1}{\sqrt{2\pi\hbar}}\wp 2\pi\cdot\,\mathrm{Ai}\big(k[x-\tfrac{E}{mg}]\big)\end{aligned}$$

which comes into precise agreement with (53.1) when we set $F\frac{1}{\sqrt{2\pi\hbar}}\wp 2\pi = \sqrt{k}$, *i.e.*, when we set $F = 1/\sqrt{2\pi\wp}$.

[25] "The free fall of quantum particles," J. of the Phys. Soc. of Japan **68**, 2543 (1999). I am indebted to Oz Bonfim for this reference. Wadati is the director of a condensed matter research group at the University of Tokyo, and in producing what is to my knowledge the first paper addressed specifically to the quantum mechanics of free fall (previous papers—uncited by him—treat the "quantum bouncer" and related systems) took his motivation from the recent explosion of interest in "falling Bose-Einstein condensates." In that macroscopic context the role of the Schrödinger equation is (we are informed) taken over by the "Gross-Pitaevskii equation," which accounts for Wadati's interest in a "hydrodynamic" formulation of his results. Wadati's familiarity with the basics of the problem appears to have been acquired from Landau & Lifshitz, *Quantum Mechanics: Non-relativistic Theory* (3rd edition 1977), §24: "Motion in a homogeneous field." But Landau & Lifshitz have in mind a homogeneous *electrical* field, so adopt a gravitationally-unnatural sign convention, which Wadati is content to live with (in his theory things fall *up*). It appears to have been the worked Problem attached to §24 that alerted Wadati to the advantages to be gained from working in the momentum representation.

• Set $f(q) = F\exp\{-\frac{s^2}{\hbar^2}q^2 - i\frac{1}{3\wp^3}q^3\}$ and, after much tedious simplification of the result reported by *Mathematica*, obtain

$$\begin{aligned}\Psi(x) &= \tfrac{1}{\sqrt{2\pi\hbar}}F\hbar\sqrt{\pi}\tfrac{1}{\sqrt{s^2[1+i(\hbar t/2ms^2)]}}\exp\Big\{-\frac{(x+\frac{1}{2}gt^2)^2}{4s^2[1+i(\hbar t/2ms^2)]}\\ &\qquad\qquad\qquad\qquad -\tfrac{i}{\hbar}mgt(x+\tfrac{1}{6}gt^2)\Big\}\\ &= F\sqrt{(\hbar/s)\sqrt{\pi/2}}\cdot\tfrac{1}{\sqrt{s(t)\sqrt{2\pi}}}\exp\Big\{-\tfrac{1}{4}\Big[\frac{x+\frac{1}{2}gt^2}{s(t)}\Big]^2 - i\varPhi(x,t)\Big\}\end{aligned}$$

If we set $F = \big[(\hbar/s)\sqrt{\pi/2}\,\big]^{-\frac{1}{2}}$—which, by the way, is exactly the value needed to achieve $\int f^*(q)f(q)\,dq = 1$—then we recover precisely the "dropped Gaussian" wavefunction of page 30, described now in physical variables. And immediately we recover the description (65.1) of $|\Psi(x,t)|^2$.

• From (74.2) it follows at $t = 0$ that

$$\Psi(x,0) = \tfrac{1}{\sqrt{2\pi\hbar}}\int f(q)\exp\Big\{i\Big[\tfrac{1}{\hbar}qx + \tfrac{1}{3\wp^3}q^3\Big]\Big\}dq$$

and therefore that

$$f(q)\exp\Big\{i\tfrac{1}{3\wp^3}q^3\Big\} = \tfrac{1}{\sqrt{2\pi\hbar}}\int \Psi(y,0)e^{-\frac{i}{\hbar}qy}\,dy$$

Returning with this information to (74.2) we have

$$\begin{aligned}\Psi(x,t) &= \tfrac{1}{2\pi\hbar}\iint \Psi(y,0)e^{-\frac{i}{\hbar}q(y-x)}\\ &\qquad\qquad\cdot\exp\Big\{i\big[-\tfrac{1}{\hbar}mgtx + \tfrac{1}{3\wp^3}(q-mgt)^3 - \tfrac{1}{3\wp^3}q^3\big]\Big\}dqdy\\ &= \sqrt{\tfrac{m}{i2\pi\hbar t}}\int \Psi(y,0)\exp\Big\{\tfrac{i}{\hbar}\big[\tfrac{m}{2t}(y-x-\tfrac{1}{2}gt^2)^2 - mgtx - \tfrac{1}{6}mg^2t^3\big]\Big\}dy\\ &= \sqrt{\tfrac{m}{i2\pi\hbar t}}\int \Psi(y,0)\exp\Big\{\tfrac{i}{\hbar}\big[\tfrac{m}{2t}(y-x)^2 - \tfrac{1}{2}mg(y+x)t - \tfrac{1}{24}mg^2t^3\big]\Big\}dy\end{aligned}$$

Wadati presents this result as a "note added in proof" which he declines to discuss because he finds it to be in some respects problematic: he appears not to appreciate that $\big[$etc.$\big]$ is just the classical action (9), and that he has in effect reconstructed the description (61) of the free fall propagator.

20. New solutions by space/time translation of old ones. If $\psi(x,t)$ describes a solution of (see again (51)) the free fall Schrödinger equation

$$\mathbf{H}\Psi - i\hbar\tfrac{\partial}{\partial t}\Psi = 0 \quad\text{with}\quad \mathbf{H} \equiv \Big\{-\tfrac{\hbar^2}{2m}\big(\tfrac{\partial}{\partial x}\big)^2 + mgx\Big\} \tag{76}$$

then so also does $\Psi(x,t) \equiv \Psi(x,t+\tau)$. The demonstration hinges on the observations that

$$\Psi(x,t) = e^{-\frac{i}{\hbar}\mathbf{H}t}\,\Psi(x,0) \;\xleftarrow[t]{\quad}\; \Psi(x,0)$$

entails

$$\boldsymbol{\Psi}(x,t) = e^{-\frac{i}{\hbar}\mathsf{H}\,t}\,\boldsymbol{\Psi}(x,0) \;\xleftarrow[t]{\qquad}\; \boldsymbol{\Psi}(x,0) = e^{-\frac{i}{\hbar}\mathsf{H}\,T}\,\Psi(x,0)$$

and that $e^{-\frac{i}{\hbar}\mathsf{H}T}$ commutes with $\frac{\partial}{\partial t}$ (this because the Hamiltonian H is t-independent). Alternatively but equivalently, one might argue from the obvious t-translation-invariance

$$K(x, t+\tau; x_0, t_0+\tau) = K(x, t; x_0, t_0)$$

of (see again (62)) the free fall propagator

$$K(x,t;x_0,t_0) = \sqrt{\tfrac{m}{2\pi i\,\hbar(t-t_0)}}\cdot e^{\frac{i}{\hbar}S(x,t;x_0,t_0)}$$

where

$$S(x,t;x_0,t_0) = \tfrac{1}{2}m\left\{\frac{(x-x_0)^2}{t-t_0} - g(x+x_0)(t-t_0) - \tfrac{1}{12}g^2(t-t_0)^3\right\}$$

We have, in the preceding paragraph, exposed the quantum counterpart of the classical circumstance described in Figure 2. The situation becomes more interesting when we look to the effect (Figure 3) of *space* translation

$$\Psi(x,t) \longmapsto \Psi(x+\mathrm{X},t) = e^{\mathrm{X}\frac{\partial}{\partial x}}\,\Psi(x,t)$$

for $e^{-\frac{i}{\hbar}\mathsf{H}T}$ and the space translation operator $e^{\mathrm{X}\frac{\partial}{\partial x}}$ do *not* commute (except in the case $g = 0$). Letting the latter act upon the Schrödinger equation (76) we obtain an equation which can be written

$$\begin{aligned}\mathsf{H}\,\Psi(x+\mathrm{X},t) &= i\hbar\big(\tfrac{\partial}{\partial t} + \tfrac{i}{\hbar}mg\mathrm{X}\big)\Psi(x+\mathrm{X},t)\\ &= i\hbar e^{-\frac{i}{\hbar}mg\mathrm{X}\,t}\tfrac{\partial}{\partial t}e^{+\frac{i}{\hbar}mg\mathrm{X}\,t}\Psi(x+\mathrm{X},t) \quad \text{by a shift rule}\end{aligned}$$

or again

$$\mathsf{H}\,\boldsymbol{\Psi}(x,t) = i\hbar\tfrac{\partial}{\partial t}\boldsymbol{\Psi}(x,t)$$
$$\boldsymbol{\Psi}(x,t) \equiv e^{\frac{i}{\hbar}mg\mathrm{X}\,t}\cdot\Psi(x+\mathrm{X},t)$$

Alternatively, one might argue from the observation that

$$K(x+\mathrm{X},t;x_0+\mathrm{X},t_0) = e^{-\frac{i}{\hbar}mg\mathrm{X}(t-t_0)}\cdot K(x,t;x_0,t_0)$$

The interesting point is that if we are to achieve "space-translational closure" within the space of free fall wavefunctions then we must expand the meaning of "translation" to include a suitably designed gauge factor.

Notice that if $\Psi(x,t)$ is normalized then both of the processes

$$\Psi(x,t) \;\longmapsto\; \boldsymbol{\Psi}(x,t) = \Psi(x,t+\tau) \tag{77.1}$$
$$\Psi(x,t) \;\longmapsto\; \boldsymbol{\Psi}(x,t) = e^{\frac{i}{\hbar}mg\mathrm{X}\,t}\cdot\Psi(x+\mathrm{X},t) \tag{77.2}$$

give rise to wavefunctions that are also normalized.

21. New solutions by Galilean transformation of old ones. Hit the Schrödinger equation (76) with the t-dependent space-translation operator $e^{vt\partial_x}$ and —noting that $e^{vt\partial_x}\partial_t = [\partial_t - v\partial_x]e^{vt\partial_x}$—obtain

$$\mathsf{H}\,\Psi(x+vt,t) = i\hbar\big(\big[\tfrac{\partial}{\partial t} - v\tfrac{\partial}{\partial x}\big] + \tfrac{i}{\hbar}mgvt\big)\Psi(x+vt,t)$$

But

$$\begin{aligned}\mathsf{H} + i\hbar v\partial_x &= -\tfrac{\hbar^2}{2m}\big[\partial_x - \tfrac{i}{\hbar}mv\big]^2 + mgx - \tfrac{1}{2}mv^2\\ &= e^{+\frac{i}{\hbar}mv\,x}\Big\{-\tfrac{\hbar^2}{2m}\partial_x^2 + mgx\Big\}e^{-\frac{i}{\hbar}mv\,x} - \tfrac{1}{2}mv^2\end{aligned}$$

so we have

$$\begin{aligned}e^{+\frac{i}{\hbar}mv\,x}&\Big\{-\tfrac{\hbar^2}{2m}\partial_x^2 + mgx\Big\}e^{-\frac{i}{\hbar}mv\,x}\Psi(x+vt,t)\\ &= i\hbar\big(\partial_t - \tfrac{i}{\hbar}\tfrac{1}{2}mv^2 + \tfrac{i}{\hbar}mgvt\big)\Psi(x+vt,t)\\ &= i\hbar e^{+\frac{i}{\hbar}\frac{1}{2}mv[vt-gt^2]}\partial_t e^{-\frac{i}{\hbar}\frac{1}{2}mv[vt-gt^2]}\Psi(x+vt,t)\end{aligned}$$

This can be written $\mathsf{H}\Psi(x,t) = i\hbar\partial_t\Psi(x,t)$ where $\Psi(x,t)$ is defined now by the process

$$\Psi(x,t) \;\longmapsto\; \Psi(x,t) \equiv e^{-\frac{i}{\hbar}[mv(x-\frac{1}{2}gt^2)+\frac{1}{2}mv^2t]}\cdot\Psi(x+vt,t) \tag{77.3}$$

To gain a clearer sense of what we have accomplished, we notice that initially

$$\Psi(x,0) = e^{-\frac{i}{\hbar}mvx}\cdot\Psi(x,0)$$

where the factor $e^{-\frac{i}{\hbar}mvx}$ serves in effect to "launch" $\Psi(x,0)$, and that free-fall propagation of $\Psi(x,0)$ produces

$$\begin{aligned}\Psi(x,t) &= \int K(x,t;y,0)\,e^{-\frac{i}{\hbar}mvy}\cdot\Psi(y,0)\,dy\\ &= \int K(x+vt,t;y,0)\,e^{-\frac{i}{\hbar}[mv(x-\frac{1}{2}gt^2)+\frac{1}{2}mv^2t]}\cdot\Psi(y,0)\,dy\\ &= e^{-\frac{i}{\hbar}[mv(x-\frac{1}{2}gt^2)+\frac{1}{2}mv^2t]}\cdot\int K(x+vt,t;y,0)\Psi(y,0)\,dy\\ &= e^{-\frac{i}{\hbar}[mv(x-\frac{1}{2}gt^2)+\frac{1}{2}mv^2t]}\cdot\Psi(x+vt,t)\\ &= e^{\frac{i}{\hbar}\Lambda(x,t)}\cdot\Psi(x+vt,t\end{aligned}$$

where $\Lambda(x,t) \equiv -mvx - \frac{1}{2}mv^2t + \frac{1}{2}mgvt^2$ is precisely the gauge term that was seen (§12) to be central to the Galilean covariance of *classical* free fall.

Again: if $\Psi(x,t)$ is normalized then so also is $\Psi(x,t)$. The transformations (77) are presented by Wadati, but without supporting discussion.

22. The free particle limit of the quantum mechanics of free fall. Trivially

$$\Big\{-\tfrac{\hbar^2}{2m}\big(\tfrac{\partial}{\partial x}\big)^2+mgx\Big\}\psi(x,t)=i\hbar\tfrac{\partial}{\partial t}\psi(x,t) \tag{51}$$

$$\downarrow$$

$$\Big\{-\tfrac{\hbar^2}{2m}\big(\tfrac{\partial}{\partial x}\big)^2\Big\}\psi(x,t)=i\hbar\tfrac{\partial}{\partial t}\psi(x,t) \quad : \quad \text{free particle Schrödinger equation}$$

and

$$K_g(x,t;x_0,0)=\sqrt{\tfrac{m}{2\pi i\hbar t}}\,\exp\Big\{\tfrac{i}{\hbar}\Big[\tfrac{m}{2t}(x-x_0)^2-\tfrac{1}{2}mg(x+x_0)t-\tfrac{1}{24}mg^2t^3\Big]\Big\} \tag{61}$$

$$\downarrow$$

$$K_0(x,t;x_0,0)=\sqrt{\tfrac{m}{2\pi i\hbar t}}\,\exp\Big\{\tfrac{i}{\hbar}\Big[\tfrac{m}{2t}(x-x_0)^2\Big]\Big\} \quad : \quad \text{free particle propagator}$$

when gravity is turned off: $g^2\downarrow 0$. More interesting is the question: What happens to the eigenfunctions? How do the Airy functions manage to become exponentials? One might adopt a strategy

$$\begin{tikzcd} K_g \arrow[r, "g^2\downarrow 0"'] & K_0 \arrow[d] \\ \text{free fall eigenfunctions} \arrow[u] & \text{free motion eigenfunctions} \end{tikzcd} \tag{78}$$

that exploits the unproblematic meaning of $K_g \longrightarrow K_0$. But there are things to be learned from the attempt to proceed directly, by asymptotic analysis. The following remarks have been abstracted from the discussion that appears on pages 48–52 of some notes already cited,[21] where I found the problem to be surprisingly ticklish.

Borrowing from page 29, we have

$$\Psi_E(x)=\sqrt{k}\cdot\mathrm{Ai}\big(k[x-a]\big)=\sqrt{k}\cdot\tfrac{1}{2\pi}\int_{-\infty}^{+\infty}e^{i\,(\frac{1}{3}u^3+k[x-a]u)}\,du$$

with

$$k\equiv\big(\tfrac{2m^2g}{\hbar^2}\big)^{\frac{1}{3}}$$

$$a\equiv\tfrac{E}{mg}\quad:\quad\text{becomes }\infty\text{ as }g\downarrow 0\text{ with }E\text{ held constant}$$

A slight notational adjustment gives

$$\Psi_g(x;E)\equiv\Psi_E(x)=\sqrt{k}\cdot\tfrac{1}{2\pi}\int_{-\infty}^{+\infty}e^{ia\,(\frac{1}{3a}u^3+k[\frac{x}{a}-1]u)}\,du$$

which by a change of variable $u\mapsto w : w^3\equiv\frac{1}{a}u^3$ becomes

$$=\tfrac{1}{2\pi\sqrt{k}}\,ka^{\frac{1}{3}}\int_{-\infty}^{+\infty}e^{ia\,(\frac{1}{3}w^3+ka^{1/3}[\frac{x}{a}-1]w)}\,dw$$

Introducing the abbreviation

$$B \equiv ka^{\frac{1}{3}} = \left[\tfrac{2m^2 g}{\hbar^2}\cdot\tfrac{E}{mg}\right]^{\frac{1}{3}} = \left[\tfrac{2mE}{\hbar^2}\right]^{\frac{1}{3}} \quad : \quad [B] = (\text{length})^{-\frac{2}{3}}$$

and noticing in this connection that $[w] = (\text{length})^{-\frac{1}{3}}$, we have

$$\Psi_g(x;E) = \tfrac{1}{2\pi\sqrt{k}}\, B\int_{-\infty}^{+\infty} e^{iah(w)}\, dw \tag{79.1}$$

with

$$h(w) \equiv \tfrac{1}{3}w^3 + B\left[\tfrac{x}{a} - 1\right]w \tag{79.2}$$

Our objective is to extract

$$\overset{\downarrow}{\Psi_0^\pm(x;E)} = \tfrac{1}{\sqrt{2\pi\hbar}}\, e^{\pm\frac{i}{\hbar}px} \quad : \quad p \equiv \sqrt{2mE} \tag{80}$$

in the limit $g^2 \downarrow 0$, but several points merit comment in advance of any such effort:

- The cases $g^2 > 0$ and $g = 0$ are in important respects *conceptually distinct*. The "natural length," "natural energy," etc. that arise (see again §14) when $g \neq 0$ are not available (become meaningless) when $g = 0$. The energy spectrum is non-degenerate when $g \neq 0$, but becomes abruptly doubly-degenerate at $g = 0$. $[\mathbf{H}, \mathbf{p}] = \mathbf{O}$ requires $g = 0$: only in the absence of gravity can p be used to identify energy eigenstates (*i.e.*, to resolve the degeneracy, which in the case $g \neq 0$ is absent).
- The functions $\Psi_g(x;E)$ and $\Psi_0^\pm(x;E)$ are dimensionally distinct

$$\begin{aligned}[]
[\Psi_g\,(x;E)] &= (\text{length})^{\frac{1}{2}-\frac{2}{3}-\frac{1}{3}} = (\text{length})^{-\frac{1}{2}}\\
[\Psi_0^\pm(x;E)] &= (\text{length}\cdot\text{momentum})^{-\frac{1}{2}}
\end{aligned}$$

so any attempt to achieve $\Psi_0^\pm(x;E) = \lim_{g^2\downarrow 0}\Psi_g(x;E)$ is—already for this reason alone—foredoomed. Look more closely to the situation. $\Psi_g(x;E)$ is co-dimensional with objects of type $(x|\Psi)$ but—since not normalizable—is not itself such an object: notated $(x|E)$, it awaits insertion into expressions of the form

$$(x|\Psi) = \int (x|E)\, dE\, (E|\Psi)$$

—the implication being that, since $[(x|E)] = (\text{length}\cdot\text{energy})^{-\frac{1}{2}}$, we *should* be looking to the asymptotics of[26]

$$(x|E)_g \equiv \tfrac{1}{2\pi\sqrt{mg}}\, B\int_{-\infty}^{+\infty} e^{iah(w)}\, dw \tag{81}$$

[26] Here I make use of $\mathcal{E}_g k = mg$.

On the other hand, $(x|p) = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}px}$ awaits insertion into expressions of the form

$$(x|\Psi) = \int (x|p)\,dp\,(p|\Psi) \quad : \quad dp = \tfrac{m}{\sqrt{2mE}}dE$$

Bearing these points in mind, and agreeing to rest content with a somewhat impressionist line of informal argument ...

We note that (81) is of such a form as to invite attack by the "method of stationary phase,"[27] which would give

$$(x|E)_g \sim \tfrac{1}{2\pi\sqrt{mg}}\, B\Big[\frac{2\pi}{ah''(w_0)}\Big]^{\frac{1}{2}} e^{i[ah(w_0)+\frac{\pi}{4}]}$$

where w_0 marks a point at which $h(w)$ assumes a locally extremal value. From

$$h'(w) = w^2 + B\big[\tfrac{x}{a} - 1\big] = 0 \quad \text{we have} \quad w_0 = \pm\sqrt{B\big[1-\tfrac{x}{a}\big]}$$

To maintain the reality of w_0 we will enforce the restriction $a > x$. From $h''(w_0) = 2w_0$ we see that w_0 marks a local minimum/maximum according as $w_0 \gtrless 0$; *i.e.*, according as we select the upper sign or the lower. We now have

$$(x|E)_g \sim \tfrac{1}{2\pi\sqrt{mg}}\, B\left[\frac{2\pi}{a2\sqrt{B\big(1-\frac{x}{a}\big)}}\right]^{\frac{1}{2}} e^{\pm i\frac{\pi}{4}}\cdot e^{iah(w_0)}$$

But *Mathematica* supplies

$$B\left[\frac{2\pi}{a2\sqrt{B\big(1-\frac{x}{a}\big)}}\right]^{\frac{1}{2}} = \sqrt{\pi}B^{\frac{3}{4}}\tfrac{1}{\sqrt{a}}\Big\{1+\tfrac{1}{4}\frac{x}{a}+\tfrac{5}{32}\big(\frac{x}{a}\big)^2+\cdots\Big\}$$

and

$$h(w_0) = \mp\tfrac{2}{3}\Big[B\big(1-\frac{x}{a}\big)\Big]^{\frac{3}{2}} = \mp B^{\frac{3}{2}}\Big\{\tfrac{2}{3}-\frac{x}{a}+\tfrac{1}{4}\big(\frac{x}{a}\big)^2+\cdots\Big\}$$

so in leading asymptotic approximation we have

$$(x|E)_g \sim \tfrac{1}{2\pi\sqrt{mga}}\sqrt{\pi}B^{\frac{3}{4}}e^{\pm i\frac{\pi}{4}}\cdot e^{\mp iaB^{\frac{3}{2}}\{\frac{2}{3}-\frac{x}{a}\}}$$

Finally use $mga = E$ and $B^{\frac{3}{2}} = \frac{\sqrt{2mE}}{\hbar} \equiv \frac{1}{\hbar}p$ to obtain

$$= \sqrt{p/2E}\cdot\tfrac{1}{\sqrt{2\pi\hbar}}\,e^{\pm\frac{i}{\hbar}px}\cdot e^{i\,(\text{phase})} \tag{82}$$

The argument, as presented, is very rough, but (if we dismiss the phase factor as an unphysical artifact) has at least led us to a result that possesses both the functional design of the free particle eigenstate $(x|p)$ and the anticipated dimensionality $(\text{length}\cdot\text{energy})^{-\frac{1}{2}}$.

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[27] See A. Erdélyi, *Asymptotic expansions* (1956), page 51. Also of special relevance is Erdélyi's §2.6.

23. Recovery of the quantum mechanics of free fall from free particle theory. The problem studied in the preceding section

$$\text{free fall eigenfunctions} \xrightarrow[g \text{ turned OFF}]{\hspace{4cm}} \text{free motion eigenfunctions}$$

(and found there to be surprisingly difficult) is of less interest to me than its inversion:

$$\text{free fall eigenfunctions} \xleftarrow[g \text{ turned ON}]{\hspace{4cm}} \text{free motion eigenfunctions}$$

The motivating questions—*How does the quantum motion of a free particle come to look "gravitational" when viewed from an accelerated frame? How do Airy functions arise transformationally from exponentials?*—clearly call for the development of a more powerful line of attack, for asymptotic analysis, since it entails the abandonment of information, cannot be carried out "in reverse."

Exploiting the well known fact that the canonical transformations of classical mechanics correspond to the unitary transformations of quantum mechanics (from which they can be recovered by passage to the classical limit), we proceed in quantum imitation of arguments developed in §11.

It proves simplest to take as our specific point of departure what on page 14 I called the "ungauged formalism." Introduce the u-parameterized family of t-dependent unitary operators

$$\mathbf{U}(t,u) \equiv e^{\frac{u}{i\hbar}\mathbf{G}(t)} \quad \text{with} \quad \mathbf{G}(t) \equiv -\tfrac{1}{2}gt^2\,\mathbf{p}$$

and construct[28]

$$\begin{aligned}
\mathbf{x}(u) &= \mathbf{U}^{-1}(t,u)\,\mathbf{x}\,\mathbf{U}(t,u)\\
&= \mathbf{x} - \tfrac{u}{i\hbar}[\mathbf{G},\mathbf{x}] + \tfrac{1}{2!}\big(\tfrac{u}{i\hbar}\big)^2[\mathbf{G},[\mathbf{G},\mathbf{x}]] - \tfrac{1}{3!}\big(\tfrac{u}{i\hbar}\big)^3[\mathbf{G},[\mathbf{G},[\mathbf{G},\mathbf{x}]]] + \cdots\\
&= \mathbf{x} - u\tfrac{1}{2}gt^2\,\mathbf{I}
\end{aligned}$$

$$\begin{aligned}
\mathbf{p}(u) &= \mathbf{U}^{-1}(t,u)\,\mathbf{p}\,\mathbf{U}(t,u)\\
&= \mathbf{p} - \tfrac{u}{i\hbar}[\mathbf{G},\mathbf{p}] + \tfrac{1}{2!}\big(\tfrac{u}{i\hbar}\big)^2[\mathbf{G},[\mathbf{G},\mathbf{p}]] - \tfrac{1}{3!}\big(\tfrac{u}{i\hbar}\big)^3[\mathbf{G},[\mathbf{G},[\mathbf{G},\mathbf{p}]]] + \cdots\\
&= \mathbf{p}
\end{aligned}$$

At $u = 0$ we recover the identity transformation, while at $u = 1$ we obtain an operator analog of (29):

$$\left.\begin{aligned}
\mathbf{x} &\longrightarrow \mathbf{x} = \mathbf{U}^{-1}\mathbf{x}\,\mathbf{U} = \mathbf{x} - \tfrac{1}{2}gt^2\,\mathbf{I}\\
\mathbf{p} &\longrightarrow \mathbf{p} = \mathbf{U}^{-1}\mathbf{p}\,\mathbf{U} = \mathbf{p}
\end{aligned}\right\} \tag{83}$$

Here I have adopted the abbreviated notation $\mathbf{U} \equiv \mathbf{U}(t) \equiv \mathbf{U}(t,1)$. And in

[28] We make use here of the operator identity at appears as (72.1) in QUANTUM MECHANICS (2000), Chapter 0. And of course, the brackets $[\bullet,\bullet]$ here signify not Poisson brackets but *commutators*. The results stated are all implications of the fundamental relation $[\mathbf{x},\mathbf{p}] = i\hbar\mathbf{I}$.

order to mimic (29)—in order to "set x and p into motion"—I have been obliged to work in what in quantum dynamics is called the "Heisenberg picture."[29] But our objectives are better served if we work in the "Schrödinger picture," as will soon emerge.

For a free particle we have (in the Schrödinger picture)

$$\mathsf{H}|\Psi) = i\hbar\partial_t|\Psi) \quad \text{with} \quad \mathsf{H} \equiv \tfrac{1}{2m}\mathsf{p}^2$$

Multiplication by U gives

$$\begin{aligned}
\mathsf{U\,H\,U}^{-1}|\psi) &= i\hbar\mathsf{U}\,\partial_t\,\mathsf{U}^{-1}|\psi) \quad \text{with} \quad |\psi) \equiv \mathsf{U}|\Psi) && (84.1)\\
&= i\hbar\big\{\partial_t + \mathsf{U}(\partial_t\,\mathsf{U}^{-1})\big\}|\psi)\\
&\downarrow\\
\mathcal{H}|\psi) &= i\hbar\partial_t|\psi) \quad \text{with} \quad \mathcal{H} \equiv \mathsf{U\,H\,U}^{-1} - i\hbar\mathsf{U}(\partial_t\,\mathsf{U}^{-1}) && (84.2)
\end{aligned}$$

But from the stipulated design of U it follows that $\mathsf{U}(\partial_t\,\mathsf{U}^{-1}) = \frac{1}{i\hbar}gt\,\mathsf{p}$, so we have

$$\begin{aligned}
\mathcal{H} &= \tfrac{1}{2m}\mathsf{p}^2 - gt\,\mathsf{p} && (85)\\
&= \tfrac{1}{2m}(\mathsf{p} - mgt\,\mathsf{I})^2 - \tfrac{1}{2}mg^2t^2\,\mathsf{I}\\
&= e^{\frac{i}{\hbar}mgt\,\mathsf{x}}\Big\{\tfrac{1}{2m}\mathsf{p}^2 - \tfrac{1}{2}mg^2t^2\,\mathsf{I}\Big\}e^{-\frac{i}{\hbar}mgt\,\mathsf{x}}
\end{aligned}$$

and the transformed free particle Schrödinger equation becomes

$$\begin{aligned}
\Big\{\tfrac{1}{2m}\mathsf{p}^2 - \tfrac{1}{2}mg^2t^2\,\mathsf{I}\Big\}e^{-\frac{i}{\hbar}mgt\,\mathsf{x}}|\psi) &= e^{-\frac{i}{\hbar}mgt\,\mathsf{x}}\,i\hbar\partial_t|\psi)\\
&= \big(i\hbar\partial_t - mg\,\mathsf{x}\big)e^{-\frac{i}{\hbar}mgt\,\mathsf{x}}|\psi)
\end{aligned}$$

Simple rearrangement gives

$$\Big\{\tfrac{1}{2m}\mathsf{p}^2 + mg\,\mathsf{x}\Big\}e^{-\frac{i}{\hbar}mgt\,\mathsf{x}}|\psi) = \big(i\hbar\partial_t + \tfrac{1}{2}mg^2t^2\big)e^{-\frac{i}{\hbar}mgt\,\mathsf{x}}|\psi)$$

---

[29] In the Heisenberg picture

$$\begin{aligned}
\text{operators move by the rule} \quad & \mathsf{A} \longrightarrow \mathsf{U}^{-1}\mathsf{A\,U}\\
\text{states move by the rule} \quad & |\psi) \longrightarrow |\psi)
\end{aligned}$$

while in the Schrödinger picture

$$\begin{aligned}
\text{operators move by the rule} \quad & \mathsf{A} \longrightarrow \mathsf{A}\\
\text{states move by the rule} \quad & |\psi) \longrightarrow \mathsf{U}|\psi)
\end{aligned}$$

Inner products *in either case* transform $(\psi|\mathsf{A}|\psi) \longrightarrow (\psi|\mathsf{U}^{-1}\mathsf{A\,U}|\psi)$.

But

$$\big(i\hbar\partial_t + \tfrac{1}{2}mg^2t^2\big) = e^{\frac{i}{\hbar}\frac{1}{6}mg^2t^3}\, i\hbar\partial_t e^{-\frac{i}{\hbar}\frac{1}{6}mg^2t^3}$$

so we arrive finally at an equation that can be written

$$\mathbf{H}|\Psi) = i\hbar\partial_t|\Psi)$$

with

$$\mathbf{H} \equiv \tfrac{1}{2m}\mathbf{p}^2 + mg\,\mathbf{x} \qquad \text{and} \qquad |\Psi) \equiv e^{-\frac{i}{\hbar}[mgt\mathbf{x} + \frac{1}{6}mg^2t^3\mathbf{I}]}\cdot \mathbf{U}|\Psi)$$

In the x-representation this becomes

$$\Big\{\tfrac{1}{2m}\big(\tfrac{\hbar}{i}\partial_x\big)^2 + mgx\Big\}\Psi(x,t) = i\hbar\partial_t\Psi(x,t) \tag{86.1}$$

with

$$\begin{aligned}\Psi(x,t) &\equiv (x|\Psi)\\ &= e^{-\frac{i}{\hbar}[mgtx + \frac{1}{6}mg^2t^3]}\cdot\Psi(x+\tfrac{1}{2}gt^2,t)\end{aligned} \tag{86.2}$$

—the assumption here being that $\Psi(x,t)$ is any solution of

$$\Big\{\tfrac{1}{2m}\big(\tfrac{\hbar}{i}\partial_x\big)^2\Big\}\Psi(x,t) = i\hbar\partial_t\Psi(x,t) \tag{86.3}$$

Equations (86) register a claim that is readily confirmed by direct calculation. The preceding argument raises several points that merit comment:

• At (32) the term gtp that enters into the design of $H(p,x)$ was attributed (*via* $\partial F_2/\partial t$) to the t-dependence of the Legendre generator F_2. At (85) the term $gt\mathbf{p}$ that enters into the similar design of $\boldsymbol{\mathcal{H}}$ is attributed (*via* $i\hbar\mathbf{U}(\partial_t\mathbf{U}^{-1})$) to the t-dependence of $\mathbf{U}$.

• It is an immediate implication of (86.2) that if $\Psi(x,t)$ is normalized then so also is free fall wavefunction $\Psi(x,t)$.

• The exponientiated expression in (86.2) has been familiar to us as a *classical* object since (on page 17) we had occasion to introduce

$$\Lambda(x,t) \equiv -mgxt - \tfrac{1}{6}mg^2t^3$$

We used *gauge arguments in the guise of shift rules* to bring about an adjustment $-gt\mathbf{p} \longrightarrow mg\mathbf{x}$ in the design of the transformed Hamiltonian.

The argument would have assumed a different appearance if we had taken the "gauged formalism" (page 17)—or better still: the "conflated formalism" (page 18)—as our point of departure. I will, in fact, look to the latter, since the exercise raises a point of analytical interest, and yields a result that will be of importance in the sequel. If we write (compare page 43)

$$\mathbf{U}(t,u) \equiv e^{\frac{u}{i\hbar}\mathbf{G}(t)} \quad \text{with } \mathbf{G} \text{ given now by} \quad \mathbf{G}(t) \equiv mgt\,\mathbf{x} - \tfrac{1}{2}gt^2\,\mathbf{p}$$

then

$$\begin{aligned}\mathbf{x}(u) &= \mathbf{U}^{-1}(t,u)\,\mathbf{x}\,\mathbf{U}(t,u)\\ &= \mathbf{x} - \tfrac{u}{i\hbar}[\mathbf{G},\mathbf{x}] + \tfrac{1}{2!}\big(\tfrac{u}{i\hbar}\big)^2[\mathbf{G},[\mathbf{G},\mathbf{x}]] - \tfrac{1}{3!}\big(\tfrac{u}{i\hbar}\big)^3[\mathbf{G},[\mathbf{G},[\mathbf{G},\mathbf{x}]]] + \cdots\\ &= \mathbf{x} - u\tfrac{1}{2}gt^2\,\mathbf{I}\end{aligned}$$

$$\begin{aligned}\mathbf{p}(u) &= \mathbf{U}^{-1}(t,u)\,\mathbf{p}\,\mathbf{U}(t,u)\\ &= \mathbf{p} - \tfrac{u}{i\hbar}[\mathbf{G},\mathbf{p}] + \tfrac{1}{2!}\big(\tfrac{u}{i\hbar}\big)^2[\mathbf{G},[\mathbf{G},\mathbf{p}]] - \tfrac{1}{3!}\big(\tfrac{u}{i\hbar}\big)^3[\mathbf{G},[\mathbf{G},[\mathbf{G},\mathbf{p}]]] + \cdots\\ &= \mathbf{p} - umgt\,\mathbf{I}\end{aligned}$$

which at $u = 1$ give (in the Heisenberg picture) the quantum analog of (41):

$$\left.\begin{aligned}\mathbf{x} \longrightarrow \mathbf{x} &= \mathbf{U}^{-1}\,\mathbf{x}\,\mathbf{U} = \mathbf{x} - \tfrac{1}{2}gt^2\,\mathbf{I}\\ \mathbf{p} \longrightarrow \mathbf{p} &= \mathbf{U}^{-1}\,\mathbf{p}\,\mathbf{U} = \mathbf{p} - mgt\,\mathbf{I}\end{aligned}\right\} \tag{83}$$

Reverting now to the Schrödinger picture, we find that

$$\mathbf{H}|\Psi) = i\hbar\partial_t|\Psi) \quad \text{with} \quad \mathbf{H} \equiv \tfrac{1}{2m}\mathbf{p}^2$$

becomes

$$\begin{aligned}\boldsymbol{\mathcal{H}}|\psi) = i\hbar\partial_t|\psi) \quad \text{with} \quad |\psi) &\equiv \mathbf{U}|\Psi)\\ \boldsymbol{\mathcal{H}} &\equiv \mathbf{U}\,\mathbf{H}\,\mathbf{U}^{-1} - i\hbar\mathbf{U}(\partial_t\mathbf{U}^{-1})\end{aligned}$$

These equations are structurally identical to (84), but $\mathbf{U}$ has now a different meaning, and we must work a bit to evaluate $\mathbf{U}(\partial_t\mathbf{U}^{-1}) = -(\partial_t\mathbf{U})\mathbf{U}^{-1}$.

MATHEMATICAL DIGRESSION: Let[30]

$$\mathbf{W} \equiv e^{\frac{i}{\hbar}(\alpha\,\mathbf{p}+\beta\,\mathbf{x})} = e^{\frac{1}{2}\frac{i}{\hbar}\alpha\beta}e^{\frac{i}{\hbar}\beta\,\mathbf{x}}e^{\frac{i}{\hbar}\alpha\,\mathbf{p}}$$

and assume the parameters α and β to be t-dependent. Then

$$\partial_t\mathbf{W} = \tfrac{i}{\hbar}\Big[\big\{\tfrac{1}{2}\partial_t(\alpha\beta) + (\partial_t\beta)\mathbf{x}\big\}\mathbf{W} + (\partial_t\alpha)e^{\frac{1}{2}\frac{i}{\hbar}\alpha\beta}e^{\frac{i}{\hbar}\beta\,\mathbf{x}}\,\mathbf{p}\,e^{\frac{i}{\hbar}\alpha\,\mathbf{p}}\Big]$$

But $e^{\frac{i}{\hbar}\beta\,\mathbf{x}}\,\mathbf{p} = (\mathbf{p} - \beta\,\mathbf{I})e^{\frac{i}{\hbar}\beta\,\mathbf{x}}$ so we have

$$\begin{aligned}\partial_t\mathbf{W} &= \mathbf{A}\,\mathbf{W}\\ \mathbf{A} &= \tfrac{i}{\hbar}\Big[\tfrac{1}{2}\partial_t(\alpha\beta) + (\partial_t\beta)\mathbf{x} + (\partial_t\alpha)(\mathbf{p} - \beta\,\mathbf{I})\Big]\\ &= \tfrac{i}{\hbar}\Big[(\partial_t\beta)\mathbf{x} + (\partial_t\alpha)\mathbf{p} + \tfrac{1}{2}\big\{\alpha(\partial_t\beta) - \beta(\partial_t\alpha)\big\}\Big]\end{aligned}$$

In the case of immediate interest $\alpha = \frac{1}{2}gt^2$ and $\beta = -mgt$, which give

$$\mathbf{U}(\partial_t\mathbf{U}^{-1}) = \tfrac{1}{i\hbar}\Big[-mg\,\mathbf{x} + gt\,\mathbf{p} + \tfrac{1}{4}mg^2t^2\,\mathbf{I}\Big]$$ END OF DIGRESSION

[30] Here I borrow freely from the early pages of Chapter 2 in QUANTUM MECHANICS (2000). See especially equation (9.1).

We therefore have

$$\begin{aligned}\mathcal{H} &= \tfrac{1}{2m}(\mathbf{p} + mgt\,\mathbf{I})^2 + mg\,\mathbf{x} - gt\,\mathbf{p} - \tfrac{1}{4}mg^2t^2\,\mathbf{I}\\ &= \tfrac{1}{2m}\,\mathbf{p}^2 + mg\,\mathbf{x} + \tfrac{1}{4}mg^2t^2\,\mathbf{I}\end{aligned}$$

giving

$$\Big\{\tfrac{1}{2m}\,\mathbf{p}^2 + mg\,\mathbf{x}\Big\}\,\mathbf{U}|\psi) = (i\hbar\partial_t - \tfrac{1}{4}mg^2t^2)\,\mathbf{U}|\psi)$$

But

$$\big(i\hbar\partial_t - \tfrac{1}{4}mg^2t^2\big) = e^{-\frac{i}{\hbar}\frac{1}{12}mg^2t^3}\,i\hbar\partial_t e^{\frac{i}{\hbar}\frac{1}{12}mg^2t^3}$$

and (drawing once again upon the operator-ordering identity that was central to the digression on the preceding page—an identity usually attributed to W. O. Kermack & W. H. McCrea (1932))

$$\begin{aligned}\mathbf{U} &= e^{\frac{1}{2}\frac{i}{\hbar}\alpha\beta}\,e^{\frac{i}{\hbar}\beta\mathbf{x}}\,e^{\frac{i}{\hbar}\alpha\,\mathbf{p}}\\ &= e^{-\frac{1}{4}\frac{i}{\hbar}mg^2t^3}\,e^{-\frac{i}{\hbar}mgt\,\mathbf{x}}\,e^{\frac{i}{\hbar}\frac{1}{2}gt^2\,\mathbf{p}}\end{aligned}$$

so in the x-representation we have

$$\begin{aligned}\Psi(x,t) &= (x|e^{\frac{i}{\hbar}\frac{1}{12}mg^2t^3}\,\mathbf{U}|\psi)\\ &= e^{\frac{i}{\hbar}\left[-mgtx + \left(\frac{1}{12}-\frac{1}{4}\right)mg^2t^3\right]}\,e^{\frac{i}{\hbar}\frac{1}{2}gt^2\left(\frac{\hbar}{i}\partial_x\right)}\,\Psi(x,t)\\ &= e^{\frac{i}{\hbar}\left[-mgtx - \frac{1}{6}mg^2t^3\right]}\,\Psi(x+\tfrac{1}{2}gt^2,t)\end{aligned}$$

Thus do we recover precisely (86). Along the way we encountered (as we encountered already on page 18) a mysterious factor $\frac{1}{4}$ but have this time been in position to watch it play its role and disappear: $\frac{1}{12} - \frac{1}{4} = -\frac{1}{6}$.

I conclude this discussion with sketches of a couple of entirely different lines of argument:

Let $\Psi(x,t)$ be—as before—any solution of the free particle Schrödinger equation

$$\tfrac{\hbar^2}{2m}\partial_x^2\Psi + i\hbar\partial_t\Psi = 0$$

Write

$$\Psi(x,t) \equiv e^{\frac{i}{\hbar}f(x,t)}\cdot\Psi(x+\tfrac{1}{2}gt^2,t)$$

and ask: What must be the structure of $f(x,t)$ if $\Psi(x,t)$ is to be a solution of the free fall equation $\frac{\hbar^2}{2m}\partial_x^2\Psi - mgx\Psi + i\hbar\partial_t\Psi = 0$? Entrusting the calculation to *Mathematica* (from whom also we borrow a non-standard device for notating partial derivatives), we find that necessarily

$$\begin{aligned}-i\hbar\big\{gt + \tfrac{1}{m}f^{(1,0)}\big\}\Psi^{(1,0)} + \big\{mgx + f^{(0,1)} + \tfrac{1}{2m}[f^{(1,0)}]^2 - i\hbar\tfrac{1}{2m}f^{(2,0)}\big\}\Psi\\ = \tfrac{\hbar^2}{2m}\Psi^{(2,0)} + i\hbar\Psi^{(0,1)}\\ = 0 \quad \text{by assumption}\end{aligned}$$

which entails

$$\begin{aligned} f^{(1,0)} + mg\,t &= 0 \\ 2m^2 g\,x + 2mf^{(0,1)} + [f^{(1,0)}]^2 - i\hbar f^{(2,0)} &= 0 \end{aligned}$$

The former condition supplies

$$f(x,t) = -mg\,x\,t + \varphi(t) \quad : \quad \varphi(t) \text{ arbitrary}$$

which when introduced into the latter condition gives $2m\varphi' + m^2g^2t^2 = 0$, the implication being that

$$\varphi(t) = -\tfrac{1}{6}mg^2t^3 + \text{arbitrary constant}$$

We are brought thus to the (independently verifiable) conclusion that if $\Psi(x,t)$ is any solution of $\frac{\hbar^2}{2m}\partial_x^2\Psi + i\hbar\partial_t\Psi = 0$ then

$$\Psi(x,t) = e^{-\frac{i}{\hbar}[mg\,x\,t + \frac{1}{6}mg^2\,t^3]} \cdot \Psi(x + \tfrac{1}{2}gt^2, t)$$

is a solution of $\frac{\hbar^2}{2m}\partial_x^2\Psi - mgx\Psi + i\hbar\partial_t\Psi = 0$.[31] It follows in particular that[32] $\Psi(x + \frac{1}{2}gt^2, t)$ itself satisfies

$$\Big\{\tfrac{1}{2m}\big(\tfrac{\hbar}{i}\partial_x - mg\,t\big)^2 + mg\,x - \big(i\hbar\partial_t + mg\,x + \tfrac{1}{2}mg^2t^2\big)\Big\}\Psi(x + \tfrac{1}{2}gt^2, t) = 0$$

which again is susceptible to immediate direct verification. Notice especially that there are here *two* references to the gravitational potential $mg\,x$, which cancel. And that after simplifications the preceding equation reads

$$\Big\{\tfrac{1}{2m}\big(\tfrac{\hbar}{i}\partial_x\big)^2 - g\,t\big(\tfrac{\hbar}{i}\partial_x\big) - i\hbar\partial_t\Big\}\Psi(x + \tfrac{1}{2}gt^2, t) = 0$$

—the validity of which is transparent.

[31] And so, of course, is $\Psi(x,t) \cdot e^{\frac{i}{\hbar}(\text{arbitrary constant})}$.

[32] We make use here of the shift rules

$$\begin{aligned} \big(\tfrac{\hbar}{i}\partial_x\big)e^{-\frac{i}{\hbar}[mg\,x\,t + \frac{1}{6}mg^2\,t^3]} &= e^{-\frac{i}{\hbar}[mg\,x\,t + \frac{1}{6}mg^2\,t^3]}\big(\tfrac{\hbar}{i}\partial_x - mg\,t\big) \\ \big(i\hbar\partial_t\big)e^{-\frac{i}{\hbar}[mg\,x\,t + \frac{1}{6}mg^2\,t^3]} &= e^{-\frac{i}{\hbar}[mg\,x\,t + \frac{1}{6}mg^2\,t^3]}\big(i\hbar\partial_t + mgx + \tfrac{1}{2}mg^2t^2\big) \end{aligned}$$

Still more simply: observe—unmotivatedly, it might appear—that from

$$S_0(x,t;x_0,t_0) = \tfrac{1}{2}m\Big\{\frac{(x-x_0)^2}{t-t_0}\Big\} \quad : \quad \text{FREE PARTICLE ACTION FUNCTION}$$

it follows that

$$\begin{aligned} S_0(x+\tfrac{1}{2}gt^2,t;x_0,t_0) &- [mgxt+\tfrac{1}{6}mg^2t^3] \\ &= \tfrac{1}{2}m\Big\{\frac{(x-x_0)^2}{t} - g(x+x_0)t - \tfrac{1}{12}g^2t^3\Big\} \\ &= S_g(x,t;x_0,0) \quad : \quad \text{FREE FALL ACTION FUNCTION} \end{aligned}$$

and therefore (see again (61)) that

$$K_g(x,t;x_0,0) = e^{-\frac{i}{\hbar}[mgxt+\frac{1}{6}mg^2t^3]}\cdot K_0(x+\tfrac{1}{2}gt^2,t;x_0,0) \tag{87}$$

which can be considered to lie at the elegant heart of (86).

24. EXAMPLE: From standing Gaussian to Gaussian in free fall. We struggled a bit in §16 to develop a description of a "dropped Gaussian wavepacket," but find ourselves in position now to accomplish that and similar tasks almost trivially. We consider it to be known[33] that if initially

$$\Psi(x,0) = \Big[\tfrac{1}{\sigma\sqrt{2\pi}}\Big]^{\frac{1}{2}}\exp\Big\{-\tfrac{1}{4}\tfrac{x^2}{\sigma^2}\Big\}$$

then by free quantum motion

$$\Psi(x,t) = \Big[\tfrac{1}{\sigma[1+i(t/\tau)]\sqrt{2\pi}}\Big]^{\frac{1}{2}}\exp\Big\{-\tfrac{1}{4}\tfrac{x^2}{\sigma^2[1+i(t/\tau)]}\Big\}$$

where $\tau \equiv 2m\sigma^2/\hbar$. The results now in hand inform us that if that initial wavepacket had been "dropped" then the evolved packet would instead be described

$$\Psi(x,t) = e^{-\frac{i}{\hbar}[mgxt+\frac{1}{6}mg^2t^3]}\cdot\Big[\tfrac{1}{\sigma[1+i(t/\tau)]\sqrt{2\pi}}\Big]^{\frac{1}{2}}\exp\Big\{-\tfrac{1}{4}\tfrac{(x+\frac{1}{2}gt^2)^2}{\sigma^2[1+i(t/\tau)]}\Big\}$$

The description (65.1) of $|\Psi(x,t)|^2$ immediately follows.

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[33] See *Gaussian wavepackets* (1998), page 4.

25. Dropped eigenfunctions. The energy spectrum of a free particle is continuous and doubly degenerate. But $[\mathbf{H}, \mathbf{p}] = \mathbf{0}$ and the spectrum of $\mathbf{p}$ is non-degenerate: it becomes natural therefore to use the momentum operator to resolve the degeneracy of the energy operator; *i.e.*, to exploit the fact that if $\mathbf{p}|p) = p|p)$ then $\mathbf{H}|p) = E|p)$ with $E = \frac{1}{2m}p^2$. In x-representation we write

$$\Psi(x,0;p) = \tfrac{1}{\sqrt{h}}\exp\Big\{\tfrac{i}{\hbar}px\Big\}$$

which when set into quantum motion becomes

$$\Psi(x,t;p) = \tfrac{1}{\sqrt{h}}\exp\Big\{\tfrac{i}{\hbar}\big[px - \tfrac{1}{2m}p^2t\big]\Big\}$$

But if such an initial function were "dropped" then, according to (86.2), we would instead obtain

$$\begin{aligned}\Psi(x,t;p) &= e^{-\frac{i}{\hbar}[mgxt + \frac{1}{6}mg^2t^3]}\cdot\tfrac{1}{\sqrt{h}}\exp\Big\{\tfrac{i}{\hbar}\big[p(x+\tfrac{1}{2}gt^2) - \tfrac{1}{2m}p^2t\big]\Big\} \qquad (88)\\ &\equiv e^{i\beta(x,t)}\cdot\tfrac{1}{\sqrt{h}}\exp\Big\{\tfrac{i}{\hbar}\big[p(x+\tfrac{1}{2}gt^2) - \tfrac{1}{2m}p^2t\big]\Big\}\end{aligned}$$

This exactly reproduces a result which at (66) we found to be in some respects perplexing (and provides yet another approach to the *derivation* of (86)). But the point of interest is that $\Psi(x,t;p)$ is $\underline{\text{not of the form } \psi(x)e^{-\frac{i}{\hbar}Et}}$:

Dropped free particle eigenfunctions are not
eigenfunctions of the free fall Hamiltonian

. . . though they are solutions of the free fall Schrödinger equation. To reproduce the buzzing free fall eigenfunctions the functions $\Psi(x,t;p)$ must be "assembled" (taken in appropriate linear combination). How is that to be accomplished?

The functions $\Psi(x,t;p)$ are readily shown to be (at every t) orthonormal and complete

$$\begin{aligned}\int \Psi^*(x,t;q)\Psi(x,t;p)\,dx &= \tfrac{1}{h}\int \exp\Big\{\tfrac{i}{\hbar}\big[(p-q)(x+\tfrac{1}{2}gt^2) - \tfrac{1}{2m}(p^2-q^2)t\big]\Big\}\\ &= \exp\Big\{\tfrac{i}{\hbar}\big[(p-q)\tfrac{1}{2}gt^2 - \tfrac{1}{2m}(p^2-q^2)t\big]\Big\}\cdot\delta(p-q)\\ &= \delta(p-q)\\ \int \Psi^*(y,t;p)\Psi(x,t;p)\,dp &= e^{i[\beta(x,t)-\beta(y,t)]}\cdot\tfrac{1}{h}\int\exp\Big\{\tfrac{i}{\hbar}p(x-y)\Big\}\,dy\\ &= e^{i[\beta(x,t)-\beta(y,t)]}\cdot\delta(x-y)\\ &= \delta(x-y)\end{aligned}$$

The implication is that solutions $\Psi(x,t)$ of the free fall Schrödinger equation can—quite generally—be written

$$\Psi(x,t) = \int c(p)\Psi(x,t;p)\,dp \qquad (89.1)$$

$$c(p) = \int \Psi^*(y,t;p)\Psi(y,t)\,dy \qquad (89.2)$$

and that so in particular can the buzzing free fall eigenfunctions[34]

$$\begin{aligned}\Psi(x,t;a) &\equiv e^{-\frac{i}{\hbar}mgat}\cdot\sqrt{k}\,\mathrm{Ai}\big(k(x-a)\big)\\ &= e^{-\frac{i}{\hbar}mgat}\cdot\sqrt{k}\,\tfrac{1}{2\pi}\int_{-\infty}^{+\infty}e^{i[k(x-a)u+\frac{1}{3}u^3]}\,du\end{aligned}$$

In the latter instance we have

$$\begin{aligned}c_a(p) &= \int e^{\frac{i}{\hbar}[mgyt+\frac{1}{6}mg^2t^3]}\cdot\tfrac{1}{\sqrt{h}}\exp\Big\{-\tfrac{i}{\hbar}\big[p(y+\tfrac{1}{2}gt^2)-\tfrac{1}{2m}p^2t\big]\Big\}\\ &\qquad\cdot\Big\{e^{-\frac{i}{\hbar}mgat}\cdot\sqrt{k}\,\tfrac{1}{2\pi}\int_{-\infty}^{+\infty}e^{i[k(y-a)u+\frac{1}{3}u^3]}\,du\Big\}\,dy\\ &= \tfrac{1}{2\pi}\sqrt{k}\cdot e^{\frac{i}{\hbar}[\frac{1}{6}mg^2t^3-\frac{1}{2}pgt^2+\frac{1}{2m}p^2t-mgat]}\\ &\qquad\cdot\int\underbrace{\Big\{\tfrac{1}{\sqrt{h}}\int e^{\frac{i}{\hbar}[mgyt-py+\hbar kyu]}\,dy\Big\}}_{\sqrt{h}\delta(\wp u-p+mgt)\quad:\quad \wp\equiv\hbar k=(2m^2g\hbar)^{\frac{1}{3}}}e^{i[-kau+\frac{1}{3}u^3]}\,du\end{aligned}$$

But $\delta(\wp u-p+mgt)=\wp^{-1}\delta\big(u-\dfrac{p-mgt}{\wp}\big)$ so the double integral becomes

$$\begin{aligned}\int\Big\{\text{etc.}\Big\}e^{i[\text{stuff}]}\,du &= \wp^{-1}\sqrt{h}\exp\Big\{i\Big[-ka\frac{p-mgt}{\wp}+\tfrac{1}{3}\big(\frac{p-mgt}{\wp}\big)^3\Big]\Big\}\\ &= \wp^{-1}\sqrt{h}\exp\Big\{\tfrac{i}{\hbar}\Big[-a(p-mgt)+\tfrac{1}{6}\frac{(p-mgt)^3}{m^2g}\Big]\Big\}\end{aligned}$$

Returning with this information to the preceding equation, and entrusting the simplifications to *Mathematica*, we obtain

$$c_a(p)=\tfrac{1}{2\pi}\wp^{-1}\sqrt{kh}\cdot e^{\frac{i}{\hbar}[-ap+\frac{1}{3}(p^3/2m^2g)]}\tag{90.1}$$

from which—miraculously, but as anticipated/required—all reference to t has disappeared. Introducing this result into the right side of (89.1) we obtain

$$\begin{aligned}&\int c(p)\Psi(x,t;p)\,dp\\ &\quad= \tfrac{1}{2\pi}\wp^{-1}\sqrt{kh}\,e^{-\frac{i}{\hbar}[mgxt+\frac{1}{6}mg^2t^3]}\\ &\qquad\cdot\tfrac{1}{\sqrt{h}}\int\exp\Big\{\tfrac{i}{\hbar}\big[p(x-a+\tfrac{1}{2}gt^2)-\tfrac{1}{2m}p^2t+\tfrac{1}{3}(p^3/2m^2g)\big]\Big\}\,dp\end{aligned}$$

A change of variables $p=\wp u$ gives

$$\begin{aligned}&= \sqrt{k}\,e^{-\frac{i}{\hbar}[mgxt+\frac{1}{6}mg^2t^3]}\\ &\qquad\cdot\tfrac{1}{2\pi}\int\exp\Big\{i\big[ku(x-a+\tfrac{1}{2}gt^2)-\tfrac{1}{2m\hbar}\wp^2u^2t+\tfrac{1}{3}u^3\big]\Big\}\,du\end{aligned}$$

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[34] See again page 35. I have elected to use a (location of the turning point) rather than $E=mga$ to label the eigenfunctions.

which at $t = 0$ simplifies markedly to yield the gratifying result

$$\begin{aligned}\int c(p)\Psi(x,0;p)\,dp &= \sqrt{k}\cdot\tfrac{1}{2\pi}\int \exp\Big\{i\big[ku(x-a)+\tfrac{1}{3}u^3\big]\Big\}\,du\\ &= \sqrt{k}\,\mathrm{Ai}\big(k(x-a)\big)\\ &= \Psi(x,0;a)\end{aligned}$$

To deal with the case $t \neq 0$ we have to work a bit: we notice that

$$\begin{aligned}&\tfrac{1}{3}u^3 - \tfrac{1}{2m\hbar}\wp^2u^2t + \tfrac{1}{2}kugt^2 - \tfrac{1}{6}mg^2t^3/\hbar\\ &\qquad = \tfrac{1}{3}u^3 - \big(\tfrac{mg^2}{2\hbar}\big)^{\frac{1}{3}}tu^2 + \big(\tfrac{mg^2}{2\hbar}\big)^{\frac{2}{3}}t^2u - \tfrac{1}{3}\big(\tfrac{mg^2}{2\hbar}\big)^{\frac{3}{3}}t^3\\ &\qquad = \tfrac{1}{3}\big[u - \big(\tfrac{mg^2}{2\hbar}\big)^{\frac{1}{3}}t\big]^3\end{aligned}$$

so if we introduce[35] $w \equiv u - \big(\tfrac{mg^2}{2\hbar}\big)^{\frac{1}{3}}t \equiv u - \omega t$ then the final equation on the preceding page can be written

$$\begin{aligned}\int c(p)\Psi(x,t;p)\,dp &= \sqrt{k}\,e^{-\frac{i}{\hbar}[mgxt]}\\ &\qquad\cdot\tfrac{1}{2\pi}\int \exp\Big\{i\big[k(w+\omega t)(x-a))+\tfrac{1}{3}w^3\big]\Big\}\,dw\\ &= e^{-\frac{i}{\hbar}[mgxt - \wp\omega t(x-a)]}\\ &\qquad\cdot\sqrt{k}\,\tfrac{1}{2\pi}\int \exp\Big\{i\big[kw(x-a))+\tfrac{1}{3}w^3\big]\Big\}\,dw\end{aligned}$$

But $\wp\omega = (2m^2g\hbar)^{\frac{1}{3}}\big(\tfrac{mg^2}{2\hbar}\big)^{\frac{1}{3}} = mg$, so

$$\begin{aligned}&= e^{-\frac{i}{\hbar}mgat}\cdot\sqrt{k}\,\mathrm{Ai}\big(k(x-a)\big)\\ &= \Psi(x,t;a)\end{aligned}$$

We have full confidence, therefore, in the accuracy of (90.1), of which

$$c_a(p) = \tfrac{1}{\sqrt{\wp}}\,e^{i\frac{1}{3}(p/\wp)^3}\,e^{-\frac{i}{\hbar}ap} \tag{90.2}$$

provides a neater description.

Notice that $g\downarrow 0$ entails $\wp \equiv (2m^2g\hbar)^{\frac{1}{3}}\downarrow 0$, and that it follows from the design of (90.2) that—consistently with our experience in §22—

$$\lim_{g\downarrow 0} c_a(p) \text{ is ill-defined}$$

On page 43 we had occasion to ask "...How do Airy functions arise transformationally from [dropped] exponentials?" An elaborately detailed

[35] Look back again to (48.4), where attention is drawn to the fact that

$$\big(\tfrac{mg^2}{2\hbar}\big)^{\frac{1}{3}} \equiv (\tau_g)^{-1} = \frac{1}{\text{gravitationally "natural time"}}$$

answer is now in hand . . . and comes, in brief, to this: *By intricate collaboration.* More technically: by t-dependent **gauged unitary transformation**.[36]

Conclusion. I have provided a fairly exhaustive account of aspects of the free fall problem which one might expect would be standard to the textbooks, but unaccountably aren't. We have seen that the classical and quantum mechanical theories display a nice formal parallelism—this not at all surprisingly, since the free fall Hamiltonian $H = \frac{1}{2m}p^2 + mgx$ depends *not worse than quadratically* upon its arguments. We have been at pains to develop the relationship between the free fall and free motion problems, *i.e.*, to establish the sense in which "weight" can be considered to be a "fictitious force, an artifact of non-inertiality." Our results serve in particular to illustrate aspects of the little-studied *general* "quantum theory of fictitious forces," but no attempt has been made here to sketch the outlines of such a general theory.

Though our results are of some independent interest (and were anticipated, so far as I am aware, only by Wadati[25]), my primary intent has been to provide a context within which to consider the "quantum bouncer" and some related systems. Those topics will be taken up in a series of companion essays.

[36] In §7 of Part C I use certain operator identities to construct an alternative derivation of our principal results that is relatively swift and painless, and that places the "gauge" aspects of the matter in a new light.